On the Oscillatory Nature of Several Fourier Multipliers

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Contents

Chapter 1

Introducción: Resumen y conclusiones

Esta tesis presenta un compendio de resultados en el área de multiplicadores de Fourier, algunos publicados en [1, 16] y otros sin publicar. En ella volveremos sobre la demostración de algunos teoremas ya conocidos, los ampliaremos y probaremos nuevos resultados dejando margen a futuras investigaciones. Pese a que el trabajo presentado es el producto de distintas colaboraciones y pueda parecer disconexo, existe un claro hilo conductor. La recurrencia de las integrales oscilatorias y la necesidad de analizarlas cuidadosamente es la clave de todas las pruebas que presentamos.

La introducción contiene una presentación de los resultados incluidos en esta tesis y cada capítulo incluye una visión más técnica del área correspondiente. Se ha intentado minimizar las repeticiones entre distintos capítulos, sin embargo, en algunos casos, estas han sido inevitables.

Para facilitar la exposición, algunos resultados obtenidos durante los estudios de doctorado no han sido incluidos en este trabajo. Especial mención merecen la solución de una conjetura de Zygmund sobre las propiedades de diferenciación de funciones sobre rectángulos [15] y un resultado sobre la localización de puntos del retículo en arcos pequeños de círculos.

Integrales oscilatorias

Los multiplicadores de Fourier en $\mathbb{R}^n$, formalmente definidos por

$$T_m f(x) := \int_{\mathbb{R}^n} m(\xi)\, \hat{f}(\xi)\, e^{2\pi i x \xi} d\xi,$$

son objeto central de estudio en el análisis armónico. La relevancia de estos operadores requiere poca introducción y se ve reflejada a través de la historia de su ejemplo más ubicuo: la transformada de Hilbert,

$$Hf(x) := \int_{\mathbb{R}} -i\mathrm{sgn}(\xi)\, \hat{f}(\xi)\, e^{2\pi i x \xi} d\xi.$$

Este operador apareció por primera vez en los trabajos de D. Hilbert en el análisis complejo y, rápidamente, se encontraron numerosas aplicaciones en diversas áreas que atrajeron la atención de la comunidad científica. En el campo del análisis armónico, la transformada de Hilbert se convirtió en el paradigma de la *integral singular* y pasó a ocupar el centro de gran parte de la investigación llevada a cabo. El estudio de estos operadores condujo al desarrollo de diversas técnicas que, incluso a día de hoy, son fundamentales: la descomposición de Calderón-Zygmund, las funciones maximales, *etc.* No obstante, en esencia, los operadores de este tipo corresponden a la integración de términos oscilatorios. Por lo tanto, para dar sentido a estas expresiones, uno podría sencillamente tratar de entender las oscilaciones y cancelaciones del integrando.

Un problema clásico consiste en entender el comportamiento asintótico de integrales de la forma

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}dx,$$

dependiendo de las propiedades de la fase ϕ. Objetos con la estructura de $I(\lambda)$ fueron posteriormente denominados por E. M. Stein en [39] como *Integrales oscilatorias del primer tipo.* Para tratar de cuantificar la cancelación del integrando es necesario entender de forma precisa la oscilación de la fase ϕ. El estudio de este fenómeno se divide normalmente en tres principios básicos que dominan el comportamiento de $I(\lambda)$: *localización*, *escalamiento* y *asintóticas.*

Probablemente el ejemplo clásico más notable de dichas integrales oscilatorias es el de las funciones de Bessel. Estas aparecen ya en los estudios pioneros de J. Fourier y para $\nu > -\frac{1}{2}$ real y $x \geq 0$ admiten la siguiente expresión:

$$J_\nu(x) := \frac{1}{\pi}\Re\int_0^\pi e^{i(\nu t - x\sin t)}dt - \frac{\sin(\pi\nu)}{\pi}\int_0^\infty e^{-x\sinh(t)-\nu t}dt.$$

Las funciones de Bessel aparecen de forma natural en la transformada de Fourier de la medida de superficie $d\sigma$ de la esfera n-dimensional:

$$\widehat{d\sigma}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}}J_{\frac{n-2}{2}}(2\pi|\xi|).$$

Sus conexiones con diversos problemas del análisis armónico despertaron gran interés en la obtención de asintóticas precisas del comportamiento de $J_\nu(x)$. Nótese que esta expresión de las funciones de Bessel nos permite visualizarlas como la suma de una integral oscilatoria dominante más un término de error. De modo que, utilizando todo el potencial de los principios de las integrales oscilatorias detallados en el Apéndice A, se obtienen las estimaciones

$$\begin{aligned} |J_\nu(x)| &\lesssim x^{-\frac{1}{2}}, \text{ cuando } x \to \infty \\ |J_\nu(x)| &\lesssim x^{\nu}, \text{ cuando } x \to 0, \end{aligned}$$

para todo $\nu > 0$ y se lleva a cabo un estudio más preciso del comportamiento de $J_\nu(x)$ cuando $x \sim \nu$. Las funciones de Bessel serán recurrentes en los Capítulos 3 and 4 ya que $\widehat{d\sigma}(\xi)$ está conectada de forma intrínseca con los multiplicadores de Fourier radiales. Exploremos esta relación en la sección siguiente.

Normas mixtas polares y cilíndricas

En el Capítulo 3 centramos nuestra atención sobre el conocido problema del multiplicador del disco, definido formalmente como

$$T_{\chi_{B(0,R)}}f(x) := \int_{\mathbb{R}^n} \chi_{B(0,R)}(\xi)\,\hat{f}(\xi)\,e^{2\pi i x\xi}d\xi.$$

Este operador corresponde al análogo esférico de las sumas parciales de la transformada de Fourier. En 1971 C. Fefferman demostró que, para $n \geq 2$, T_{χ_B} no es un operador acotado en $L^p(\mathbb{R}^n)$ para toda $p \neq 2$, [23]. La atención de la comunidad matemática se dirigió entonces al estudio del análogo esférico de las sumas de Césaro-Féjer para integrales de Fourier: los multiplicadores de Bochner Riesz. Pese a que se ha progresado mucho en el estudio de estos operadores, tan solo el caso 2-dimensional es, a día de hoy, bien comprendido. Esto es debido a que están conectados intrínsecamente con los conjuntos de Kakeya y todavía se deben salvar, en dimensiones superiores, muchas dificultades geométricas [8, 12].

Sin embargo, en 1989, A. Córdoba [13] y G. Mockenhaupt [30] mostraron independientemente que

$$T_{\chi_{B(0,R)}} : L^p_{rad}L^2_{ang}(\mathbb{R}^n) \to L^p_{rad}L^2_{ang}(\mathbb{R}^n),$$

para todo $\frac{2n}{2+1} < p < \frac{2n}{n-1}$. Ambas pruebas se basan en la estructura *polar* del espacio de normas mixtas $L^p_{rad}L^2_{ang}(\mathbb{R}^n)$, correspondiente al espacio de funciones que satisfacen que

$$\left(\int_0^\infty r^{n-1}\left(\int_{\mathbb{S}^{n-1}} |f(r\theta)|^2\,d\sigma(\theta)\right)^{\frac{p}{2}} dr\right)^{\frac{1}{p}} < \infty.$$

Retomamos con mayor generalidad la prueba de A. Córdoba y de ella emergen nuevos operadores de la forma

$$T^s f(r\theta) = \sum_{k,j} Y_k^j(\theta)\int_0^\infty f_{k,j}(t)\,t^{\frac{n+2k-1}{2}} r^{-\frac{n+2k-1}{2}} k_\alpha(r,t,s)\,dt, \tag{1.0.1}$$

donde (r,θ) corresponde a las coordenadas polares en $\mathbb{R}^n$, $s \in (0,\infty)$ y

$$\begin{aligned} k_\alpha(t,r,s) &= s\frac{\sqrt{t}J'_\alpha(ts)\,J_\alpha(rs)\sqrt{r}}{2(t-r)} + s\frac{\sqrt{t}J'_\alpha(ts)\,J_\alpha(rs)\sqrt{r}}{2(t+r)}. \\ &\quad + s\frac{\sqrt{t}J_\alpha(ts)\,J'_\alpha(rs)\sqrt{r}}{2(r-t)} + s\frac{\sqrt{t}J'_\alpha(ts)\,J_\alpha(rs)\sqrt{r}}{2(t+r)}. \end{aligned} \tag{1.0.2}$$

Apoyándonos en el trabajo de J. A. Barceló [2] y en las estimadas de las funciones de Bessel (2.0.3), demostramos que estos operadores están acotados uniformemente en los espacios de norma mixta polar.

Proposition 1.1. *Sea $f \in L^p_{rad}L^2_{ang}(\mathbb{R}^n)$, entonces para toda $\frac{2n}{n+1} < p < \frac{2n}{n-1}$*

$$\|T^s f\|_{p,2} \leq C_{p,n}\,\|f\|_{p,2}, \tag{1.0.3}$$

donde la constante $C_{p,n}$ es uniforme en s.

Una consecuencia inmediata de este resultado es el hecho, previamente demostrado por J. Duoandikoetxea *et al.* [19], de que todo operador T_m asociado a un multiplicador radial m de soporte compacto y variación acotada está también acotado en el mismo rango de normas mixtas polares. A continuación, adaptamos para operadores T^s la desigualdad con pesos introducida en *[3]* por Carbery *et al.,* y producimos el siguiente teorema de tipo Marcinkiewicz:

Theorem 1.1 (A. Córdoba, E. L-C)**.** *Sea T_m un multiplicador radial de Fourier, definido formalmente por*

$$T_m f(x) = \int_{\mathbb{R}^n} m(|\xi|) \hat{f}(\xi) e^{2\pi i x \xi} d\xi,$$

con multiplicador $m \in L^\infty(\mathbb{R})$ que para cada intervalo diádico I satisface,

$$\int_I |m'(t)| \, dt \leq C, \tag{1.0.4}$$

uniformemente en I. Entonces T_m extiende a un operador acotado en $L^p_{rad} L^2_{ang}(\mathbb{R}^n)$ si y sólo si $\frac{2n}{n+1} < p < \frac{2n}{n-1}$.

Continuamos extendiendo la teoría de multiplicadores radiales de Fourier a sólidos de revolución en dimensiones superiores. Sea S un sólido de revolución alrededor de la dirección zenital (o última coordenada) de $\mathbb{R}^{n+1}$.

$$S = \left\{(r\theta, z) \in \mathbb{R}^{n+1},\ \theta \in S^{n-1},\ -a \leq z \leq a,\ 0 \leq r \leq g(z) \in C(a,b)\right\}, \tag{1.0.5}$$

con $g(z) > 0$ para casi todo $z \in [-a, a]$.

Theorem 1.2 (A. Córdoba, E. L-C)**.** *El multiplicador de Fourier T_{χ_S} asociado al un sólido revolución S de la forma (1.0.5) extiende a un operador acotado en $L^p_{rad} L^2_{ang} L^2_{zen}(\mathbb{R}^{n+1})$, si y sólo si $\frac{2n}{n+1} < p < \frac{2n}{n-1}$.*

El espacio de funciones mencionado en este teorema corresponde al espacio de normas mixtas *cilíndrico.* En $\mathbb{R}^{n+1}$, con $n \geq 2$, consideramos coordenadas cilíndricas (r, θ, z) donde las primeras componentes (r, θ) corresponden a las coordenadas polares estándar en $\mathbb{R}^n$; $0 < r < \infty,\ \theta \in S^{n-1}$, y $z \in \mathbb{R}$ denota la coordenada zenital.

En este sistema de coordenadas, la norma $L^p_{rad} L^2_{zen} L^2_{ang}(\mathbb{R}^{n+1})$ viene dada por

$$\|f\|_{p,2,2} := \left(\int_0^\infty r^{n-1} \left(\int_{-\infty}^\infty \int_{S^{n-1}} |f(r,\theta,z)|^2 \, d\sigma(\theta) \, dz\right)^{\frac{p}{2}} dr\right)^{\frac{1}{p}}. \tag{1.0.6}$$

Este espacio normado lo encontramos también en la esencia del Capítulo 4, donde tratamos ciertas cuestiones relacionadas con la famosa conjetura de restricción. Propuesta por E. M. Stein, esta conjetura afirma que la restricción de la transformada de Fourier de una función integrable f a la esfera unidad,

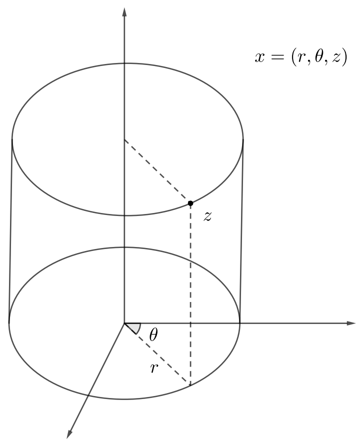

Figure 1.0.1: Coordenadas cilíndricas

$\hat{f}|_{S^{n-1}}$, define un operador acotado de $L^p(\mathbb{R}^n)$, $n \geq 2$, a $L^q\left(S^{n-1}\right)$ siempre y cuando

$$1 \leq p < \frac{2n}{n+1}, \; \frac{1}{q} \geq \frac{n+1}{n-1}\left(1 - \frac{1}{p}\right). \tag{1.0.7}$$

Esta conjetura ha sido completamente verificada en dimensión $n = 2$ por C. Fefferman [23], y tan sólo son conocidos ciertos resultados parciales en dimensiones superiores. La validez de la conjetura en el rango $q = 2$ y $1 \leq p \leq \frac{2(n+1)}{n+3}$ es el mejor resultado conocido y fue obtenido de forma independiente por P. Tomas y E. M. Stein [47].

Pese a que existen muchas publicaciones de diversos autores que arrojan cierta luz sobre la conjetura de restricción, su validez permanece totalmente abierta en dimensiónes $n \geq 3$. Uno de los progresos más sorprendentes aparece en la tesis doctoral de B. Barcelo [**?**], en la que demostró que el resultado de C. Fefferman tamibén se satisface para el cono en $\mathbb{R}^3$. Este resultado ha sido generalizado a dimensiones 4 por T. Wolff [51] y recientemente a dimensión 5 por Y. Ou [31]. Otro resultado interesante fue dado por L. Vega en su tesis doctoral [48], donde obtuvo el rango completo de la conjetura de restricción en el caso que $q = 2$ si se substituye el espacio $L^p(\mathbb{R}^n)$ por $L^p_{rad}L^2_{ang}(\mathbb{R}^n)$. Recientemente este resultado ha atraído cierta atención dada su relación con la ecuación de Schrödinger esféricamente promediada, vease [46].

A continuación consideramos la restricción de la transformada de Fourier a otras superficies de revolución en espacios de norma mixta cilíndrica. Nótese que se han tratado algunos casos previamente [28, 29], pero presentamos un procedimiento más general válido para todas las superficies de revolución compactas C^1,

$$\Gamma = \left\{(g(z), \theta, z) \in \mathbb{R}^{n+1}, \; \theta \in S^{n-1}, \; a \leq z \leq b, \; 0 \leq g \in C^1(a, b)\right\}.$$

Theorem 1.3 (A. Córdoba, E. L-C.)**.** *Sea Γ una superficie de revolución compacta, entonces la restricción de la transformada de Fourier a Γ es un operador* *acotado de* $L^p_{rad}L^2_{zen}L^2_{ang}\left(\mathbb{R}^{n+1}\right)$ *a* $L^2(\Gamma)$*, i.e. existe una constante* C_p *que satisface*

$$\left(\int_a^b\int_{S^{n-1}} g(z)^{n-1}\sqrt{1+g'(z)^2}\left|\hat{f}(g(z),\theta,z)\right|^2 d\theta dz\right)^{\frac{1}{2}} \leq C_p\|f\|_{L^p_{rad}L^2_{zen}L^2_{ang}(\mathbb{R}^{n+1})}, \quad (1.0.8)$$

siempre que $1\leq p<\frac{2n}{n+1}$.

La transformada de Hilbert con una fase oscilatoria

Cambiemos de tema y fijemos nuestra atención en la teoría clásica de las ecuaciones diferenciales parabólicas, desarrollada en la decada de los años 60. Consideremos la ecuación parabólica más básica en $\mathbb{R}\times[0,\infty)$

$$\frac{du}{dt}-\frac{d^2u}{dx^2}=f,$$

para cierta función f. Si f es “buena”, valdría con soporte compacto y Hölder continua, entonces podemos derivar una expresión de Duhamel de las soluciones de esta ecuación,

$$u(x,t)=\int_0^t\int_{\mathbb{R}}\Phi(x-\xi,t-\zeta)f(\xi,\zeta)d\xi d\zeta,$$

donde Φ es la solución fundamental de la ecuación del calor

$$\Phi(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ix\xi-t\xi^2}d\xi.$$

Para estudiar problemas de convergencia al dato de frontera de las soluciones, se necesita un buen control de la segunda derivada espacial de u. Para ello aprovechamos la estructura de convolución de dichas soluciones y obtenemos que

$$\frac{d^2u}{dx^2}(x,t):=\int_0^t\int_{\mathbb{R}}K(x-\xi,t-\zeta)f(\xi,\zeta)d\xi d\zeta,$$

con

$$K(x,t)=\Phi(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi)^2e^{ix\xi-t\xi^2}d\xi.$$

Es una tarea sencilla comprobar que este núcleo satisface la siguiente propiedad de homogeneidad:

$$K\left(\alpha x,\alpha^2t\right)=\alpha^{-3}K(x,t).$$

N. Rivière observó que esta condición de homogeneidad mixta era crucial a la hora de imitar el *método de rotaciones* desarrollado por P. Calderón y A. Zygmund. De hecho, con gran generalidad, dado un núcleo impar de dos variables reales $K(x,y)$,

$$K(-x,-y) = -K(x,y),$$

con homogeneidad parabólica,

$$K\left(\alpha x, \alpha^2 y\right) = \alpha^{-3} K(x,y),$$

podemos considerar su operador de convolución asociado

$$Tf(x,y) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-u, y-v) K(u,v)\, dudv.$$

Entonces, el método de rotaciones adaptado a las coordenadas parabólicas, $u = r\cos\theta$ y $v = r^2\sin\theta$, produce la siguiente expresión del operador:

$$Tf(x,y) = \frac{1}{2}\int_0^{2\pi} \Omega(\theta) K(\cos\theta, \sin\theta) \int_{-\infty}^{\infty} \frac{1}{r} f\left(x - r\cos\theta, y - r^2\sin\theta\right) drd\theta,$$

donde Ω es suave y simétrica en el círculo. Entonces, si $K \in L^1\left(\mathbb{S}^1\right)$, podemos aplicar la desigualdad de Minkowski y obtener la estimación

$$\|Tf\|_p \lesssim \int_0^{2\pi} \left\|\mathcal{H}^\theta f\right\|_p,$$

donde

$$\mathcal{H}^\theta f := \int_{-\infty}^{\infty} \frac{1}{r} f\left(x - r, y - c_\theta r^2\right) dr. \tag{1.0.9}$$

Claramente, para conseguir un control L^p sobre T es suficiente mostrar que

$$\left\|\mathcal{H}^\theta f\right\|_p \lesssim \|f\|_p$$

uniformemente en el coeficiente c_θ.

Mediante teorema de Plancherel es relativamente sencillo ver que $\left\|\mathcal{H}^\theta\right\|_{2\to 2} \sim \sup_\lambda \left\|T_\lambda^\theta\right\|_{2\to 2}$, donde $\lambda \in \mathbb{R}$ y

$$T_\lambda^\theta f(x) := \int_{\mathbb{R}} \frac{e^{2\pi i\lambda c_\theta s^2}}{s} f(x-s)\, ds. \tag{1.0.10}$$

Dado que T_λ^θ es de nuevo un operador de convolución, aplicando una vez más el teorema de Plancherel derivamos que

$$\left\|T_\lambda^\theta f\right\|_2^2 = \int_{\mathbb{R}^2} \left|\hat{f}(\xi)\right|^2 \left|\int_{\mathbb{R}} \frac{e^{2\pi i\left(\lambda c_\theta s^2 + \xi s\right)}}{s} ds\right|^2 d\xi.$$

Por lo tanto, la deseada propiedad de acotación en L^2 de T_λ^θ se garantiza siempre que el segundo integrando sea uniformemente acotado en ξ y λc_θ. Este acercamiento fue posteriormente generalizado a polinomios generales P de una variable real. Dicho problema consiste en obtener acotaciones de

$$\left|\int_{\mathbb{R}} \frac{e^{2\pi i P(s)}}{s} ds\right| \tag{1.0.11}$$

que sean uniformes en los coeficientes de P.

E. Fabes [21] empleó el *método del gradiente* para obtener esta deseada acotación en el caso donde $P(s) = a_1 s + a_2 s^2$. Sin embargo, para poder aplicar este método es necesario conocer con precisión el conjunto de ceros de P. Por tanto, si se quisiera abordar nuestro problema mediante estas técnicas, sería necesario controlar de forma uniforme el conjunto de ceros del operador P, lo que parece una tarea imposible. Fueron E. M. Stein y S. Wainger [42] quienes entendieron que el lema de van der Corput es suficiente para resolver este problema. La prueba de dicha solución ha sido incluida en el Apéndice A.2.

Los operadores $\mathcal{H}^\theta$ y T_λ^θ admiten generalizaciones a dimensiones superiores siguiendo construcciones similares a las que han sido detalladas para el caso 1-dimensional. La *transformada de Hilbert doble en* $\mathbb{R}^3$ *sobre una superficie polinomial* P se define formalmente como

$$\mathcal{H}^P f(x,y,z) := \int_{\mathbb{R}^2} f(x-s, y-t, z-P(s,t)) \frac{dsdt}{st}, \tag{1.0.12}$$

y la *transformada de Hilbert doble generalizada en* $\mathbb{R}^2$ *con una fase polinomial* P como

$$T_\lambda^P f(x,y) := \int_{\mathbb{R}^2} \frac{e^{i\lambda P(x-s,y-t)}}{(x-s)(y-t)} f(s,t)\, dsdt, \tag{1.0.13}$$

donde $\lambda \in \mathbb{R}$, y P es un polinomio de dos variables

$$P(x,y) := \sum_{(i,j)\in\Delta} c_{i,j} x^i y^j,$$

que satisface $P(0,0) = 0$ y $\nabla P(0,0) = 0$, y Δ denota el conjunto de índices del polinomio con coeficiente distinto de cero. Nótese que (1.0.12) y (1.0.13) difieren crucialmente de $\mathcal{H}^\theta$, T_λ^θ y de sus operadores uni-paramétricos análogos en $\mathbb{R}^n$ considerados en [34, 35], ya que nos situamos en el marco multi-paramétrico.

Como es de esperar, la teoría $L^p(\mathbb{R}^2)$ de T_λ^P puede ser deducida del operador más complejo $\mathcal{H}^P$, estudiado en su versión local por A. Carbery *et al.* [5] y en su versión general por S. Patel [33]. De hecho, para relacionar el comportamiento de ambos operadores se necesita de una uniformidad extra en λ. Su relación es la siguiente:

$$\sup_\lambda \left\|T_\lambda^P\right\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)} \leq \left\|\mathcal{H}^P\right\|_{L^p(\mathbb{R}^3)\to L^p(\mathbb{R}^3)}, \tag{1.0.14}$$

y

$$\sup_\lambda \left\|T_\lambda^P\right\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \sim \left\|\mathcal{H}^P\right\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)}. \tag{1.0.15}$$

Sin embargo, las propiedades de acotación de estos operadores no son tan sencillas como en el caso de un solo parámetro. Sorprendentemente, el comportamiento de estos operadores multi-paramétricos está relacionado con ciertas propiedades geométricas del diagrama de Newton de P. Mediante una combinación del teorema principal de S. Patel en [33] y (1.0.14), se obtiene el siguiente corolario:

Sea $\mathcal{C}$ la clausura convexa de Δ en $\mathbb{R}^2$ y

$$\mathcal{D} := \{(i,j) \in \Delta, (i,j) \text{ es una esquina (vértice) de } \mathcal{C}\}.$$

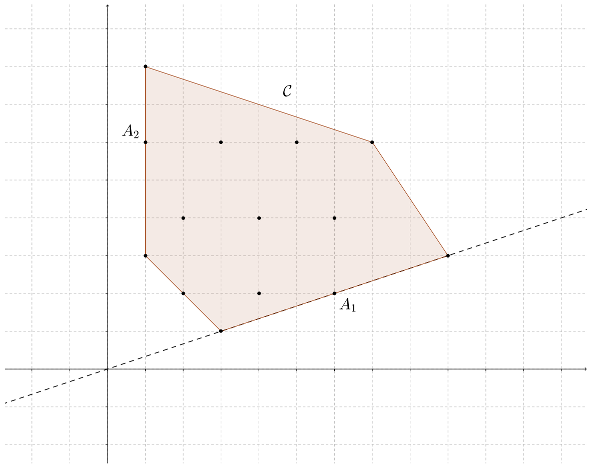

Figure 1.0.2: En este ejemplo A_1 debe satisfacer la condición de paridad, mientras que A_2 puede tener ambas coordenadas impares.

Se dice que un polinomio es admisible si y sólo si $P(0,0) = 0$, $\nabla P(0,0) = 0$, para cada $(m,n) \in \mathcal{D}$ al menos uno, m o n, es par, y además si alguna arista (extendida) de $\mathcal{C}$ pasa por el origen (a lo sumo existen dos aristas) entonces sus vértices correspondientes deben de tener al menos una coordenada par.

Corollary 1.1 (S. Patel, 2008)**.** *Sea T_λ^P la transformada de Hilbert generalizada asociada a una fase polinomial P. Entonces dado $1 < p < \infty$,*

$$\sup_\lambda \left\| T_\lambda^P f \right\|_{L^p(\mathbb{R}^2)} \le C \left\| f \right\|_{L^p(\mathbb{R}^2)}$$

para toda $f \in L^p\left(\mathbb{R}^2\right)$ si y sólo si P pertenece a la clase de polinomios admisibles.

Por otro lado, como consecuencia de la teoria multi-paramétrica desarrollada por F. Ricci y E. M. Stein en [36], es posible obtener uniformidad en los coeficientes de P imponiendo ciertas condiciones de paridad extra sobre nuestro polinomio:

Se dice que *el polynomio P está en $\mathcal{P}_{even}$* si para cada $(m,n) \in \Delta$, al menos uno m o n es par.

Corollary 1.2 (S. Patel, 2008)**.** *Sea T_λ^P la transformada de Hilbert generalizada asociada a una fase polinomial P. Entonces dado $1 < p < \infty$,*

$$\sup_\lambda \left\| T_\lambda^P f \right\|_{L^p(\mathbb{R}^2)} \le C \left\| f \right\|_{L^p(\mathbb{R}^2)}$$

uniformemente en los coeficientes de P, para toda $f \in L^p\left(\mathbb{R}^2\right)$, si y sólo si $P \in \mathcal{P}_{even}$.

El último resultado presentado en esta tesis atañe el comportamiento en espacios extremos de T_λ^P. Recalquemos que un polinomio $P(x,y)$ no tiene términos lineales en ninguna variable si $\partial_x P(0,y) = 0$ para todo $y \in \mathbb{R}$ y

$\partial_y P(x,0) = 0$ para todo $x \in \mathbb{R}$, es decir que P no contiene terminos de la forma xy^j o $x^i y$. Tabmién decimos que un polynomio no tiene términos puros en ninguna variable si no contiene términos de la forma x^i o y^j.

Theorem 1.4 (O. Bakas, E. L-C., J. Wright)**.** *Sea T_λ^P la transformada de Hilbert generalizada asociada a una fase polinomial P en la clase de polinomios admisibles que no contenga términos lineales o puros en ninguna variable. Entonces, para toda $f \in H^1_{\text{rect}}(\mathbb{R}^2)$,*

$$\sup_\lambda \left\| T_\lambda^P f \right\|_{L^1(\mathbb{R}^2)} \leq C \left\| f \right\|_{H^1_{\text{rect}}(\mathbb{R}^2)}.$$

Además, una sencilla modificación nos permite implementar el resultado de F. Ricci y E. M. Stein en el siguiente corolario:

Corollary 1.3. *Sea T_λ^P la transformada de Hilbert generalizada asociada a una fase polinomial $P \in \mathcal{P}_{even}$ en la clase de polinomios admisibles que no contenga términos lineales opuros en ninguna variable. Entonces, para toda $f \in H^1_{\text{rect}}(\mathbb{R}^2)$,*

$$\sup_\lambda \left\| T_\lambda^P f \right\|_{L^1(\mathbb{R}^2)} \leq C \left\| f \right\|_{H^1_{\text{rect}}(\mathbb{R}^2)}$$

uniformemente en los coeficientes de P.

Aquí el espacio $H^1_{\text{rect}}(\mathbb{R}^2)$ denota el espacio rectangular de Hardy usual en $\mathbb{R}^2$, que se define a continuación.

Definition 1.1. Se dice que $f \in H^1_{\text{rect}}(\mathbb{R}^2)$ si f admite una descomposición rectangular. Es decir, que existe una secuencia de escalares $(c_{I\times J})_{I\times J}$ sumable y una secuencia de átomos $a_{I\times J}$ tal que $f = \sum_{I\times J} c_{I\times J} a_{I\times J}$ en el sentido de las distribuciones, donde I y J son intervalos diádicos en $\mathbb{R}$ y $a_{I\times J}$ satisface

1. $\operatorname{supp}(a_{I\times J}) \subset I \times J$.
2. $\int_I a_{I\times \mathrm{J}}(x,y)\, dx = 0$, para todo $y \in J$.
3. $\int_J a_{I\times J}(x,y)\, dy = 0$, para todo $x \in I$.
4. $\|a_{I\times J}\|_{L^2} \leq (|I|\,|J|)^{-\frac{1}{2}}$.

Se define la norma H^1_{rec} de una función f como

$$\|f\|_{H^1_{\text{rect}}} = \inf \sum_{I\times J} |c_{I\times J}|,$$

donde el ínfimo se toma sobre todas las posibles descomposiciones diádicas $f = \sum_{I\times J} c_{I\times J} a_{I\times J}$.

Nótese que el resultado clásico de L. Carleson [7] muestra que $H^1_{\text{rect}}(\mathbb{R}^2)$ es un subespacio propio del espacio producto de Hardy $H^1_{\text{prod}}(\mathbb{R}^2)$. El espacio producto de Hardy $H^1_{\text{prod}}(\mathbb{R}^2)$ se define como el espacio de todas las funciones integrables en $\mathbb{R}^2$ tales que $H_1(f)$, $H_2(f)$, $H_1 \otimes H_2(f) \in L^1(\mathbb{R}^2)$, donde H_i denota la transformada de Hilbert en la variable $i = 1, 2$. Para más detalles sobre espacios de Hardy multi-paramétricos, vease [10].

Probaremos este resultado considerando primero el caso en el que P contiene un único monomio, es decir cuando $P(s,t) = c_{i,j}x^iy^j$ con $i,j \geq 1$ e $i \times j$ par. Mediante este acercamiento tratamos de identificar las dificultades subyacentes en el caso general y a la par elucidar la prueba. El procedimiento que seguiremos en el capítulo 5 es reminiscente del llevado a cabo por F. Ricci y E. M. Stein [34], pese a que diversos tecnicismos propios del caso multi-linear deben ser salvaguardados. De hecho, la dificultad principal consiste en obtener acotaciones de la integral oscilatoria

$$\int_{\mathbb{R}^2} \frac{e^{i\lambda\left\{(x-s)^i(y-t)^j-(x-u)^i(y-v)^j\right\}}}{(x-s)(y-t)(x-u)(y-v)} G(x,y,s,t,u,v)\,dxdy,$$

donde G es una función de corte suave, problema que está intrínsecamente ligado con la integral oscilatoria (1.0.11).

Investigaciones futuras

Los resultados presentados en esta tesis no cierran ninguna área de investigación, sino más bien insinuan posibles desarrollos a explorar. En esta sección queremos presentar algunos de ellos que parecen alcanzables en un futuro próximo. En particular, junto con O. Bakas y J. Wright, estamos realizando progresos en una extensión de un resultado de E. M. Stein y S. Wainger para el caso 2-paramétrico.

No obstante, vamos a presentar primero algunas extensiones de los resultados obtenidos en los capítulos 3 y 4, relacionados con la teoría de los espacios de norma mixta.

Continuando con las ideas que están tras la prueba del Teorema 1.2 surge de forma natural una cuestión sobre las propiedades de acotación del multiplicador asociado a un solido de revolución T_S. La esperanza es que en $\mathbb{R}^{n+1}$ se pueda acotar una función de Littlewood-Paley en cada corte de $\mathbb{R}^n$ por medio del Teorema 1.1 y entonces se pueda integrar en la dirección zenital.

Insistamos de nuevo en que las estimaciones de restricción en normas mixtas han sido recientemente el tema de varias publicaciones. Por un lado, Z. Guo [26] y C. Miao *et al.* [29] han implementado estimaciones de restricción sobre normas mixtas cilíndricas $L^p_{rad}L^2_{ang}L^2_{zen}(\mathbb{R}^n)$ en el estudio de la ecuación de Schrödinger esféricamente promediada. Por tanto, podriamos tratar de implementar nuestras estimaciones de restricción obtenidas, para el estudio de ecuaciones esférícamente promediadas cuyo espectro de Fourier esté adaptado a otras superficies de revolución. Por otra parte, en una publicación muy reciente, E. Carneiro *et al. [9]* han estudiado los extremizadores en las estimaciones de restricción para la esfera, obtenidas por L. Vega in [48]. Otra posible línea de trabajo sería la extensión de este resultado a las estimaciones de restricción adaptadas a otras superficies de revolución.

Los contenidos del capítulo 5 permiten presentar problemas abiertos más interesantes, que merecen una discusión más extensa. Recientemente en trabajo conjunto con O. Bakas y J. Wright hemos conseguido extender el Teorema 1.4 y el Corolario 1.3 a polynomios que contengan términos lineales en cualquier variable, por ejemplo $P(x,y) = xy^2$. Sin embargo todavía no sabemos controlar polynomios que tengan terminos puros en alguna de las variables, por ejemplo $P(x,y) = x^2 + x^3y^2$.

Por otro lado, recordemos el siguiente resultado de R. Fefferman, [24]:

Proposition 1.2. *Sea T un operador lineal acotado en $L^2\left(\mathbb{R}^2\right)$. Además, supongamos que para cualquier dilatación $\gamma > 2$ existe $\epsilon > 0$ fijo y una constante C_ϵ tal que*

$$\int_{(\gamma R)^c} |T a_R(x)|\, dx < C_\epsilon \gamma^{-\epsilon}, \tag{1.0.16}$$

para cualquier átomo rectangular a_R adaptado al rectángulo R. Entonces T extiende a un operador acotado de $H^1_{prod}\left(\mathbb{R}^2\right)$ a $L^1\left(\mathbb{R}^2\right)$.

Nótese que aunque en la prueba del Teorema 1.4 demostramos que

$$\int_{R^c} \left|T_\lambda^P a_R(x)\right| dx < C$$

y que, por tanto, el operador $T_\lambda^P : H^1_{rec}\left(\mathbb{R}^2\right) \to L^1\left(\mathbb{R}^2\right)$, nuestro método no permite obtener el decaimiento extra detallado en (1.0.16). De hecho en una futura publicación mostraremos que para todo átomo rectangular a_R con R centrado en el origen, $\nexists \epsilon > 0$ tal que

$$\int_{(\gamma R)^c} \left|T_\lambda^P a_R(x)\right| dx < C_\epsilon \gamma^{-\epsilon},$$

para toda $\gamma > 2$. El estudio de las propiedades de acotación de T_λ^P en el espacio $H^1_{prod}\left(\mathbb{R}^2\right)$ toma un nuevo interés ya que claramente muestra un comportamiento diferenciado del observado en $H^1_{rec}\left(\mathbb{R}^2\right)$. Por tanto la pregunta permanece abierta:

¿Es T_λ^P un operador acotado entre $H^1_{prod}\left(\mathbb{R}^2\right)$ y $L^1\left(\mathbb{R}^2\right)$?

Existe otro problema similar de carácter más compejo:

¿Es T_λ^P un operador acotado entre $L\log L^k\left(\mathbb{R}^2\right)$ y $L^{1,\infty}\left(\mathbb{R}^2\right)$ para algún $k \geq 1$?

Otra curiosidad que surge de forma natural en este estudio es la existencia de un teorema de interpolación entre $H^1_{rec}\left(\mathbb{R}^2\right)$ y $L^2\left(\mathbb{R}^2\right)$. No tenemos constancia de dicho resultado en la literatura, pero sí de la existencia de técnicas de interpolación entre $H^1_{prod}\left(\mathbb{R}^2\right)$ y $L^2\left(\mathbb{R}^2\right)$. Si semejante resultado fuese cierto, entonces recuperar la teoría L^p general sería una tarea sencilla.

Observemos también que podemos tratar de obtener ciertos análogos en dimensiones superiores del Corolario 1.3 ya que el Corolario 1.2 sigue siendo válido en estos casos.

Finalmente recalquemos que el comportamiento del operador $\mathcal{H}^P$ es mucho más complejo que el de T_λ^P y que, por tanto, todas las preguntas análogas para este operador siguen abiertas, incluyendo la validez del Teorema 1.4.

Chapter 2

Introduction: Summary and conclusions

This thesis presents a compendium of results, some published in [1, 16] and some unpublished, around the subject of Fourier multiplier operators. We revisit some old theorems, extend them and produce new results with room for further research. Although the work presented is the product of different collaborations and may arguably be disconnected, there is an undeniable underlying thread connecting it. The recurrence of oscillatory integrals and the need to carefully analyze their behaviour is the key to every proof that follows.

The introduction presents an overlook of the results included in this thesis and each chapter will have a more detailed introduction that stands on its own. Although much effort has been put into avoiding repetitions amongst different chapters, the reader might find some redundancies across the text.

For the sake of exposition, several results obtained during the doctoral training have not been included in this thesis. These include the solution to a conjecture of Zygmund on the differentiability properties of functions over rectangles [15] and a result on the location of lattice points in small arcs of a circle (Theorem 1, [16]).

Oscillatory integrals

Fourier multiplier operators in $\mathbb{R}^n$, formally defined by

$$T_m f(x) := \int_{\mathbb{R}^n} m(\xi) \hat{f}(\xi) e^{2\pi i x\xi} d\xi, \tag{2.0.1}$$

are a central object of study in the area of harmonic analysis. The relevance of these operators needs little introduction and may be reflected by the story of its most ubiquitous example: the Hilbert transform,

$$Hf(x) := \int_{\mathbb{R}} -i\operatorname{sgn}(\xi) \hat{f}(\xi) e^{2\pi i x\xi} d\xi.$$

This transformation first appeared in the work of D. Hilbert in the area of complex analysis. It soon found many applications in partial differential equations and signal processing and attracted the attention of the broad scientific community. In the area of harmonic analysis the Hilbert transform became the paradigm of a *singular integral* and was placed at the center this research area. The study of these operators led to the development of many techniques central to harmonic analysis such as the Calderón-Zygmund decomposition, maximal functions, etc. In its essence, however, operators of this type correspond to the integration of oscillatory terms. Therefore, in order to make sense of these expressions one may simply try to understand the oscillation and cancellations of the integrand.

A classical problem is that of understanding the asymptotic behaviour of integrals of the form

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}dx,$$

depending on the properties of the phase ϕ. Objects of the form $I(\lambda)$ were later referred by E. M. Stein in [39] as *oscillatory integrals of the first kind.* In order to quantify the cancellation of the integrand, one needs to understand precisely how the phase ϕ oscillates. The study of this phenomena is generally split into three principles that rule the behaviour of $I(\lambda)$: *localization, scaling* and *asymptotic.*

Probably the most notable classical example of these oscillatory integrals are Bessel functions, which date back to the early work of J. Fourier, and admit the following expression for real $\nu > -\frac{1}{2}$ and $x \geq 0$:

$$J_\nu(x) := \frac{1}{\pi}\Re\int_0^\pi e^{i(\nu t - x\sin t)}dt - \frac{\sin(\pi\nu)}{\pi}\int_0^\infty e^{-x\sinh(t)-\nu t}dt. \qquad (2.0.2)$$

Bessel functions arise naturally as the Fourier transform of the surface measure $d\sigma$ of the n-dimensional sphere S^{n-1} :

$$\widehat{d\sigma}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}}J_{\frac{n-2}{2}}(2\pi|\xi|).$$

Its connections with many problems in harmonic analysis have put much emphasis in obtaining precise asymptotics of $J_\nu(x)$. Note that this expression of Bessel functions allows us to understand it as a dominating oscillatory integral and an error term. Therefore the full potential of the oscillatory integral principles and techniques detailed in Appendix A are used to obtain the asymptotics

$$\begin{aligned} |J_\nu(x)| &\lesssim x^{-\frac{1}{2}}, \text{ as } x \to \infty \\ |J_\nu(x)| &\lesssim x^{\nu}, \text{ as } x \to 0, \end{aligned} \qquad (2.0.3)$$

for $\nu > 0$, and to carefully study the behaviour of $J_\nu(x)$ when $x \sim \nu$. Bessel functions will be prevalent in Chapters 3 and 4 as $\widehat{d\sigma}(\xi)$ is intricately connected to the Fourier transform of radial multipliers. We further explore this relationship in the next section.

Polar and cylindrical mixed norms

In Chapter 3 we center our attention around the well-known ball multiplier, defined formally as

$$T_{\chi_{B(0,R)}}f(x) := \int_{\mathbb{R}^n}\chi_{B(0,R)}(\xi)\hat{f}(\xi)e^{2\pi ix\xi}d\xi.$$

This operator corresponds to the spherical analogue of the partial summation operator for the Fourier transform. In 1971 C. Fefferman showed that, for $n \geq 2$, T_{χ_B} does not bound $L^p(\mathbb{R}^n)$ to itself for any $p \neq 2$, [23]. The attention of the mathematical community then veered towards the study the spherical analogue of the Cesàro-Fejér sums for Fourier integrals: the Bochner Riesz multipliers. Although much progress has been made in the study of these operators only the 2-dimensional case is well understood as it is intricately connected with Kakeya sets and many geometrical difficulties need to be overcome in higher dimensions, [8, 12].

However, in 1989, A. Córdoba [13] and G. Mockenhaupt [30] showed independently that

$$T_{\chi_{B(0,R)}} : L^p_{rad}L^2_{ang}(\mathbb{R}^n) \to L^p_{rad}L^2_{ang}(\mathbb{R}^n),$$

for all $\frac{2n}{2+1} < p < \frac{2n}{n-1}$. Both proofs rely on the *polar* structure of the *mixed norm space* $L^p_{rad}L^2_{ang}(\mathbb{R}^n)$ corresponding to the space of functions satisfying

$$\left(\int_0^\infty r^{n-1}\left(\int_{\mathbb{S}^{n-1}} |f(r\theta)|^2 \, d\sigma(\theta)\right)^{\frac{p}{2}} dr\right)^{\frac{1}{p}} < \infty.$$

We revisit A. Córdoba's proof and bring to light new operators of the form

$$T^s f(r\theta) = \sum_{k,j} Y_k^j(\theta) \int_0^\infty f_{k,j}(t)\, t^{\frac{n+2k-1}{2}} r^{-\frac{n+2k-1}{2}} k_\alpha(r,t,s)\, dt, \tag{2.0.4}$$

where (r,θ) correspond to polar coordinates in $\mathbb{R}^n$, $s \in (0,\infty)$ and

$$\begin{aligned} k_\alpha(t,r,s) &= s\frac{\sqrt{t}J'_\alpha(ts)\,J_\alpha(rs)\sqrt{r}}{2(t-r)} + s\frac{\sqrt{t}J'_\alpha(ts)\,J_\alpha(rs)\sqrt{r}}{2(t+r)}. \\ &\quad + s\frac{\sqrt{t}J_\alpha(ts)\,J'_\alpha(rs)\sqrt{r}}{2(r-t)} + s\frac{\sqrt{t}J'_\alpha(ts)\,J_\alpha(rs)\sqrt{r}}{2(t+r)}. \end{aligned} \tag{2.0.5}$$

Leaning on J. A. Barceló's work [2] and the Bessel functions estimates (2.0.3), these operators will be shown to be uniformly bounded in the same range of polar mixed norm spaces.

Proposition 2.1. *Let $f \in L^p_{rad}L^2_{ang}(\mathbb{R}^n)$, then for every $\frac{2n}{n+1} < p < \frac{2n}{n-1}$*

$$\|T^s f\|_{p,2} \leq C_{p,n} \|f\|_{p,2}, \tag{2.0.6}$$

where the constant $C_{p,n}$ is uniform in s.

A consequence of this result is the fact, previously shown by J. Duoandikoetxea *et al.* [19], that any operator T_m with a compactly supported radial multiplier m of bounded variation is also bounded in the same range of polar mixed norm spaces. We then adapt, to operators T^s, the weighted inequality showed in *[3]* by Carbery *et al.,* and produce the following Marcinkiewicz type theorem:

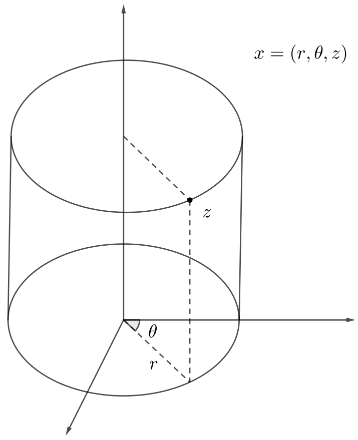

Figure 2.0.1: Cylindrical coordinates

Theorem 2.1 (A. Córdoba, E. L-C)**.** *Let T_m be a radial Fourier multiplier operator defined as formally as*

$$T_m f(x) = \int_{\mathbb{R}^n} m(|\xi|)\, \hat{f}(\xi)\, e^{2\pi i x\xi} d\xi$$

with a multiplier $m \in L^\infty(\mathbb{R})$ and for each dyadic interval I,

$$\int_I |m'(t)|\, dt \leq C, \tag{2.0.7}$$

uniformly in I. Then T_m extends to a bounded operator in $L^p_{rad} L^2_{ang}(\mathbb{R}^n)$ if and only if $\frac{2n}{n+1} < p < \frac{2n}{n-1}$.

We further extend the theory of radial Fourier multipliers to solids of revolution in "higher dimensions". Let S be a solid of revolution around zenithal (or last) coordinate of $\mathbb{R}^{n+1}$,

$$S = \left\{(r\theta, z) \in \mathbb{R}^{n+1},\ \theta \in S^{n-1},\ -a \leq z \leq a,\ 0 \leq r \leq g(z) \in C(a,b)\right\}, \tag{2.0.8}$$

with $g(z) > 0$ for almost every $z \in [-a, a]$.

Theorem 2.2 (A. Córdoba, E. L-C)**.** *Then the Fourier multiplier operator T_{χ_S} associated to a solid of revolution (2.0.8) extends to a bounded operator of $L^p_{rad} L^2_{ang} L^2_{zen}\left(\mathbb{R}^{n+1}\right)$, if and only if $\frac{2n}{n+1} < p < \frac{2n}{n-1}$.*

The space of functions mentioned in this theorem corresponds to the *cylindrical* mixed norm space. In $\mathbb{R}^{n+1}$, with $n \geq 2$, we consider cylindrical coordinates (r, θ, z) where the first components (r, θ) correspond to the standard polar coordinates in $\mathbb{R}^n$; $0 < r < \infty$, $\theta \in S^{n-1}$, and $z \in \mathbb{R}$ denotes the zenithal coordinate.

In this coordinate system, the $L^p_{rad}L^2_{zen}L^2_{ang}\left(\mathbb{R}^{n+1}\right)$ norm is given by

$$\|f\|_{p,2,2} := \left(\int_0^\infty r^{n-1}\left(\int_{-\infty}^{\infty}\int_{S^{n-1}}|f(r,\theta,z)|^2\,d\sigma(\theta)\,dz\right)^{\frac{p}{2}}dr\right)^{\frac{1}{p}}. \tag{2.0.9}$$

This space of functions is also at the center of Chapter 4, where some questions related to the well-known restriction conjecture are addressed. First proposed by E. M. Stein, the conjecture asserts that the restriction of the Fourier transform of a given integrable function f to the unit sphere, $\hat{f}|_{S^{n-1}}$, defines a bounded operator from $L^p(\mathbb{R}^n)$, $n \geq 2$, to $L^q\left(S^{n-1}\right)$ so long as

$$1 \leq p < \frac{2n}{n+1},\ \frac{1}{q} \geq \frac{n+1}{n-1}\left(1-\frac{1}{p}\right). \tag{2.0.10}$$

This conjecture has been fully proved in dimension $n = 2$ by C. Fefferman [23] and only partial results are known in higher dimensions. The validity of the conjecture in the range $q = 2$ and $1 \leq p \leq \frac{2(n+1)}{n+3}$ is the best known result and was obtained independently by P. Tomas and E. M. Stein [47].

Although there are many interesting publications by several authors casting some light on the restriction conjecture, its proof remains open in dimension $n \geq 3$. One of the more remarkable improvements was B. Barcelo's thesis [2]. He proved that Fefferman's result also holds for the cone in $\mathbb{R}^3$. Another interesting result was given by L. Vega in his Ph.D. thesis [48], where he obtained the best full range of the restriction inequality for $q = 2$ when the space $L^p(\mathbb{R}^n)$ is replaced by $L^p_{rad}L^2_{ang}(\mathbb{R}^n)$. This result has recently attracted much attention due to its connections with the spherically averaged Schrödinger equation, cf. [46].

Here we shall consider the restriction of the Fourier transform to other surfaces of revolution in these mixed norm spaces. Several special cases have already been treated [28, 29] but we present a more general and unified proof for C^1 compact surfaces of revolution

$$\Gamma = \left\{(g(z),\theta,z) \in \mathbb{R}^{n+1},\ \theta \in S^{n-1},\ a \leq z \leq b,\ 0 \leq g \in C^1(a,b)\right\}.$$

Theorem 2.3 (A. Córdoba, E. L-C.)**.** *Let Γ be a compact surface of revolution, then the restriction of the Fourier transform to Γ is a bounded operator from $L^p_{rad}L^2_{zen}L^2_{ang}\left(\mathbb{R}^{n+1}\right)$ to $L^2(\Gamma)$, i.e. there exists a finite constant C_p such that*

$$\left(\int_a^b\int_{S^{n-1}} g(z)^{n-1}\sqrt{1+g'(z)^2}\left|\hat{f}(g(z),\theta,z)\right|^2 d\theta dz\right)^{\frac{1}{2}} \leq C_p\|f\|_{L^p_{rad}L^2_{zen}L^2_{ang}(\mathbb{R}^{n+1})}, \tag{2.0.11}$$

so long as $1 \leq p < \frac{2n}{n+1}$.

Hilbert transforms with an oscillatory phase

We completely change the subject and go back to a classical theory developed in the 1960s around the theory of parabolic partial differential equations. Consider the most basic parabolic equation in $\mathbb{R}\times[0,\infty)$

$$\frac{du}{dt}-\frac{d^2u}{dx^2}=f,$$

for some function f. If f is "good", say with compact support and Hölder continuous, then we may derive a Duhamel expression of the solutions of this equation,

$$u\left(x,t\right)=\int_0^t\int_{\mathbb{R}}\Phi\left(x-\xi,t-\zeta\right)f\left(\xi,\zeta\right)d\xi d\zeta,$$

where Φ is the fundamental solution of the heat equation

$$\Phi\left(x,t\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ix\xi-t\xi^2}d\xi.$$

To study problems of convergence of the solutions in the boundary, a good control of the second spatial derivative of u is needed. However, exploiting the convolution nature of the solutions we obtain that

$$\frac{d^2u}{dx^2}\left(x,t\right):=\int_0^t\int_{\mathbb{R}}K\left(x-\xi,t-\zeta\right)f\left(\xi,\zeta\right)d\xi d\zeta,$$

with

$$K\left(x,t\right)=\Phi\left(x,t\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(i\xi\right)^2e^{ix\xi-t\xi^2}d\xi.$$

It is a simple matter to check that this kernel satisfies the following homogeneity property:

$$K\left(\alpha x,\alpha^2t\right)=\alpha^{-3}K\left(x,t\right).$$

N. Rivière observed that this mixed homogeneity condition was crucial when trying to imitate the *method of rotations* developed by P. Calderón and A. Zygmund. In fact, with great generality, given an odd kernel $K\left(x,y\right)$ of two real variables,

$$K\left(-x,-y\right)=-K\left(x,y\right),$$

with parabolic homogeneity,

$$K\left(\alpha x,\alpha^2y\right)=\alpha^{-3}K\left(x,y\right),$$

we may consider its associated convolution operator

$$Tf\left(x,y\right):=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f\left(x-u,y-v\right)K\left(u,v\right)dudv.$$

Then the method of rotations adapted to the parabolic coordinates $u=r\cos\theta$ and $v=r^2\sin\theta$ yields

$$Tf\left(x,y\right)=\frac{1}{2}\int_0^{2\pi}\Omega\left(\theta\right)K\left(\cos\theta,\sin\theta\right)\int_{-\infty}^{\infty}\frac{1}{r}f\left(x-r\cos\theta,y-r^2\sin\theta\right)drd\theta,$$

where Ω is smooth and symmetric in the circle. Therefore, if $K\in L^1\left(\mathbb{S}^1\right)$,we may apply Minkowski's inequality and obtain

$$\left\|Tf\right\|_p\lesssim\int_0^{2\pi}\left\|\mathcal{H}_\theta f\right\|_p,$$

where

$$\mathcal{H}^{\theta} f := \int_{-\infty}^{\infty} \frac{1}{r} f\left(x-r, y-c_{\theta} r^{2}\right) dr. \tag{2.0.12}$$

Clearly, to gain an L^p control of T it is enough to show that

$$\left\|\mathcal{H}^{\theta} f\right\|_{p} \lesssim \|f\|_{p}$$

uniformly in the coefficient c_θ.

By means of Plancherel's theorem it is easy to see that $\left\|\mathcal{H}^{\theta}\right\|_{2\to 2} \sim \sup_\lambda \left\|T_\lambda^{\theta}\right\|_{2\to 2}$, where $\lambda \in \mathbb{R}$ and

$$T_{\lambda}^{\theta} f(x) := \int_{\mathbb{R}} \frac{e^{2\pi i \lambda c_{\theta} s^{2}}}{s} f(x-s)\, ds. \tag{2.0.13}$$

Since T_λ^θ is a convolution operator, again Plancherel's theorem yields

$$\left\|T_{\lambda}^{\theta} f\right\|_{2}^{2} = \int_{\mathbb{R}^{2}} \left|\hat{f}(\xi)\right|^{2} \left| \int_{\mathbb{R}} \frac{e^{2\pi i\left(\lambda c_{\theta} s^{2}+\xi s\right)}}{s} ds \right|^{2} d\xi.$$

Therefore, the desired L^2 boundedness of T_λ^θ is ensured as long as the inner integrand is uniformly bounded in ξ ad λ. This approach was later generalized to general real polynomials P of one variable. The problem is to find bounds of the oscillatory integral

$$\left| \int_{\mathbb{R}} \frac{e^{2\pi i P(s)}}{s} ds \right| \tag{2.0.14}$$

that are uniform in the coefficients of P.

E. Fabes [21] employed the *method of steepest descent* to obtain this cherished bound in the original case where $P(s) = a_1 s + a_2 s^2$. However, in order to employ this method it is necessary to know precisely the zero set of P. Therefore, to use this approach for general polynomials of higher degree, one would have to control uniformly its zero set; which seems an impossible task. It was E. M. Stein and S. Wainger [42] who understood that it is enough to use van der Corput's lemma to obtain the desired bound. We have included the proof of this theorem in the Appendix A.2.

The operators $\mathcal{H}^\theta$ and T_λ^θ admit generalizations in higher dimensions following similar constructions as those detailed in the one dimensional case. The *double Hilbert transform in* $\mathbb{R}^3$ *along a polynomial surface* P is defined formally as

$$\mathcal{H}^{P} f(x, y, z) := \int_{\mathbb{R}^{2}} f(x-s, y-t, z-P(s, t)) \frac{ds dt}{st}, \tag{2.0.15}$$

and the *generalized double Hilbert transform in* $\mathbb{R}^2$ *with a polynomial phase* P as

$$T_{\lambda}^{P} f(x, y) := \int_{\mathbb{R}^{2}} \frac{e^{i \lambda P(x-s, y-t)}}{(x-s)(y-t)} f(s, t)\, ds dt, \tag{2.0.16}$$

where $\lambda \in \mathbb{R}$, and P is a polynomial of two real variables

$$P(x, y) := \sum_{(i, j) \in \Delta} c_{i, j} x^{i} y^{j},$$

satisfying $P(0,0) = 0$ and $\nabla P(0,0) = 0$, and Δ denotes the set of indices of non-zero coefficients of the polynomial. Note that these operators differ crucially

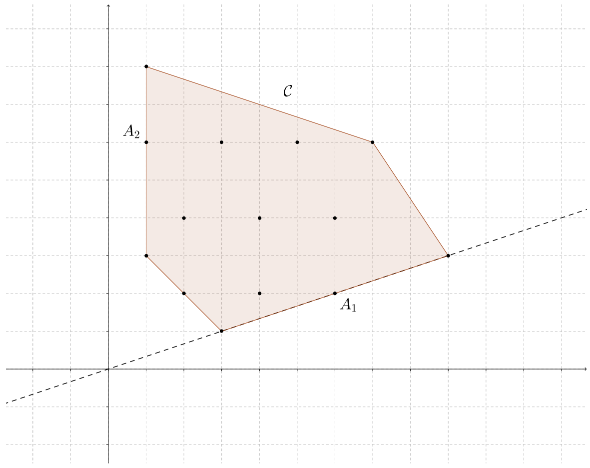

Figure 2.0.2: In this example A_1 must satisfy the parity condition, whereas A_2 may have both coordinates odd

from $\mathcal{H}^\theta$ and T_λ^θ, and their analogous one-parameter operators in $\mathbb{R}^n$ considered in [34, 35], as we place ourselves in the multi-parameter setting.

As expected, the $L^p\left(\mathbb{R}^2\right)$ theory of T_λ^P can be deduced from that of the more complicated operator $\mathcal{H}^P$, studied for the localized operator by A. Carbery *et al.* [5] and for the global operator by S. Patel [33]. In fact, in order to relate both operators, an extra uniformity in λ is needed. The relation is as follows

$$\sup_\lambda \left\|T_\lambda^P\right\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)} \leq \left\|\mathcal{H}^P\right\|_{L^p(\mathbb{R}^3)\to L^p(\mathbb{R}^3)}, \tag{2.0.17}$$

and

$$\sup_\lambda \left\|T_\lambda^P\right\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \sim \left\|\mathcal{H}^P\right\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)}. \tag{2.0.18}$$

However, the boundedness properties of these operators is not as simple as in the one parameter case. Surprisingly, the behaviour of these higher multi-parameter operators is related to the geometric properties of the Newton diagram of P. A consequence of the main theorem of S. Patel in [33] and (2.0.17), yields the following Corollary:

Let $\mathcal{C}$ be the closed convex hull of Δ in $\mathbb{R}^2$ and let

$$\mathcal{D} := \{(i,j) \in \Delta, (i,j) \text{ is a corner point (vertex) of } \mathcal{C}\}.$$

We say that a polynomial is admissible if $P(0,0) = 0$, $\nabla P(0,0) = 0$, for each $(m,n) \in \mathcal{D}$, at least one of m and n is even, and furthermore if any (extended) edge of $\mathcal{C}$ passes through the origin (there are at most two such edges) then every point on that edge must have at least one even coordinate.

Corollary 2.1 (S. Patel, 2008)**.** *Let T_λ^P be the operator associated to the polynomial P. Then for any $1 < p < \infty$,*

$$\sup_\lambda \left\|T_\lambda^P f\right\|_{L^p(\mathbb{R}^2)} \leq C \left\|f\right\|_{L^p(\mathbb{R}^2)},$$

for all $f \in L^p(\mathbb{R}^n)$, if and only if P belongs to the class of admissible polynomials.

Also, as a consequence of the multi-parameter theory developed by F. Ricci and E. M. Stein in [36], it is possible to further obtain uniformity in the coefficients of P if extra parity conditions are imposed:

We say that the *polynomial P is in $\mathcal{P}_{even}$* if for every $(m,n) \in \Delta$, at least one of m or n is even.

Corollary 2.2 (F. Ricci, E. M. Stein, 1992)**.** *Let T_λ^P be the operator associated to the polynomial P. Then for any $1 < p < \infty$,*

$$\sup_\lambda \left\|T_\lambda^P f\right\|_{L^p(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)}$$

uniformly in the coefficients of P, for any $f \in L^p(\mathbb{R}^n)$, if and only if $P \in \mathcal{P}_{even}$.

The following result we present in this thesis concerns the endpoint behaviour of the operator T_λ^P. Let us highlight that we say that a polynomial $P(x,y)$ does not contain linear terms in either variable if $\partial_x P(0,y)$ for all $y \in \mathbb{R}$ and $\partial_y P(x,0)$ for all $x \in \mathbb{R}$, or equivalently that it does not contain terms xy^j or $x^i y$. Further, we say that a polynomial does not contain pure terms in either variable if it does not contain terms of the form x^i or y^j.

Theorem 2.4 (O. Bakas, E. L-C., J. Wright)**.** *Let T_λ^P be the operator associated to an admissible polynomial P without pure or linear terms in either variable. Then for every $f \in H^1_{\text{rect}}(\mathbb{R}^2)$,*

$$\sup_\lambda \left\|T_\lambda^P f\right\|_{L^1(\mathbb{R}^2)} \leq C \|f\|_{H^1_{\text{rect}}(\mathbb{R}^2)}.$$

Furthermore, after a simple modification we may implement F. Ricci and E. M. Stein's corollary in the following result:

Corollary 2.3. *Let T_λ^P be the operator associated to an admissible polynomial $P \in \mathcal{P}_{even}$ without pure or linear terms in either variable. Then for every $f \in H^1_{\text{rect}}(\mathbb{R}^2)$,*

$$\sup_\lambda \left\|T_\lambda^P f\right\|_{L^1(\mathbb{R}^2)} \leq C \|f\|_{H^1_{\text{rect}}(\mathbb{R}^2)}$$

uniformly in the coefficients of P.

Here the space $H^1_{\text{rect}}(\mathbb{R}^2)$ denotes the usual rectangular Hardy space in $\mathbb{R}^2$ defined as follows.

Definition 2.1. We say that $f \in H^1_{\text{rect}}(\mathbb{R}^2)$ if f admits a rectangular decomposition. That is, there exists a summable sequence of scalars $(c_{I\times J})_{I\times J}$ and a sequence of rectangular atoms $a_{I\times J}$ such that $f = \sum_{I\times J} c_{I\times J} a_{I\times J}$ in the sense of distributions, where I and J are dyadic intervals in $\mathbb{R}$ and $a_{I\times J}$ satisfies

1. $\operatorname{supp}(a_{I\times J}) \subset I \times J$.
2. $\int_I a_{I\times \mathrm{J}}(x,y)\,dx = 0$, for all $y \in J$.
3. $\int_J a_{I\times J}(x,y)\,dy = 0$, for all $x \in I$.
4. $\|a_{I\times J}\|_{L^2} \leq (|I|\,|J|)^{-\frac{1}{2}}$.

Moreover, the H^1_{rec} norm of a function f is defined as

$$\|f\|_{H^1_{\text{rect}}} = \inf \sum_{I\times J} |c_{I\times J}|,$$

where the infimum is taken over all possible rectangular decompositions of $f = \sum_{I\times J} c_{I\times J} a_{I\times J}$.

Remark that a classical result of Carleson [7] shows that $H^1_{\text{rect}}\left(\mathbb{R}^2\right)$ is a proper subspace of the product Hardy space $H^1_{\text{prod}}\left(\mathbb{R}^2\right)$. The product Hardy space $H^1_{\text{prod}}\left(\mathbb{R}^2\right)$ is defined as the space of all integrable functions f on $\mathbb{R}^2$ such that $H_1(f)$, $H_2(f)$, $H_1 \otimes H_2(f) \in L^1(\mathbb{R}^2)$, where H_i denotes the Hilbert transform in the i-th variable, $i = 1, 2$. For more details on multi-parameter Hardy spaces, see [10].

This result will be proved by first considering the single monomial case, that is when $P\left(s,t\right) = c_{i,j}x^i y^j$, with $i, j \geq 1$ and $i \times j$ even. The hope is that this approach will identify the difficulties underlying the general case and elucidate the proof. The procedure followed throughout the proof is similar to that of F. Ricci and E. M. Stein [34] even though many technicalities particular to the multilinear case have to be overcome. In fact, the main challenge that needs to be addressed is to find a bound of the oscillatory integral

$$\int_{\mathbb{R}^2} \frac{e^{i\lambda\left\{(x-s)^i(y-t)^j-(x-u)^i(y-v)^j\right\}}}{(x-s)\,(y-t)\,(x-u)\,(y-v)} G\left(x,y,s,t,u,v\right) dxdy,$$

where G is a smooth cutoff function, which is undeniably connected to the oscillatory integral presented in (2.0.14).

Further work

The results presented in this thesis do not close any area of investigation, but they rather hint some possible extensions to be explored. In this section we would like to present a number of them that we believe are attainable in the near future. In particular, in joint work with O. Bakas and J. Wright we are making progress in a remarkable question extending a result of E. M. Stein and S. Wainger to the two-parametric case.

Let us, however, first present some extensions of the results obtained in Chapters 3 and 4 related to the theory of mixed norms.

Following the ideas behind the proof of Theorem 2.2 arises a natural question on the boundedness properties of the multiplier associated to a solid of revolution T_S. The hope is that in $\mathbb{R}^{n+1}$ we may bound a Littlewood-Paley square function in each slice of $\mathbb{R}^n$ by means of Theorem 2.1 and then integrate in the zenithal direction.

Let us highlight that restriction estimates in mixed normed spaces have recently been the subject of several recent publications. On one hand Z. Guo [26] and C. Miao *et al.* [29] have been implementing cylindrical mixed norms $L^p_{rad}L^2_{ang}L^2_{zen}\left(\mathbb{R}^n\right)$ restriction estimates to the study of the spherically averaged Schrödinger and wave equations. Therefore another possible extension of our work is the implementation of the restriction estimates obtained to the study of spherically averaged equations with a Fourier spectrum adapted to other surfaces of revolution. On another hand, in a very recent publication E. Carneiro *et al.* [9] study the extremizers of the restriction estimate over the sphere obtained by L. Vega in [48]. A possible line of work would be to extend this new result to the extremizers of the restriction estimate adapted to other surfaces of revolution.

The content of Chapter 5 allows for more interesting further work that I want to discuss a bit more at length. In recent work with O. Bakas and J. Wright we have managed to extend Theorem 2.4 and Corollary 2.3 to polynomials containing linear terms in either variable. However, we still don't know how to deal with polynomials containing pure terms in either variable, say $P(x,y) = x^2 + x^3y^2$.

On another hand recall the following result of R. Fefferman, [24]:

Proposition 2.2. *Let T be a linear operator bounded on $L^2(\mathbb{R}^2)$. Further, suppose that for any dilation $\gamma > 2$ there exists some fixed $\epsilon > 0$ and a constant C_ϵ such that*

$$\int_{(\gamma R)^c} |T a_R(x)|\, dx < C_\epsilon \gamma^{-\epsilon}, \tag{2.0.19}$$

for every rectangular atom a_R adapted to the rectangle R. Then T extends to a bounded operator from $H^1_{prod}(\mathbb{R}^2)$ to $L^1(\mathbb{R}^2)$.

Note that although in the proof of Theorem 2.4 we show that

$$\int_{R^c} \left|T^P_\lambda a_R(x)\right| dx < C$$

and thus that the operator $T^P_\lambda : H^1_{rec}(\mathbb{R}^2) \to L^1(\mathbb{R}^2)$, our method does not seem to be able to obtain the extra decay detailed in (2.0.19). In fact in a future publication we will show that for any rectangular atom a_R centered in 0 and for any dilation $\gamma > 2$, $\nexists \epsilon > 0$ and a constant C_ϵ such that

$$\int_{(\gamma R)^c} \left|T^P_\lambda a_R(x)\right| dx < C_\epsilon \gamma^{-\epsilon}.$$

The study of the behaviour of T^P_λ in $H^1_{prod}(\mathbb{R}^2)$ thus takes new interest as it clearly shows a distinct behaviour from that observed in $H^1_{rec}(\mathbb{R}^2)$. Hence, the question remains open:

Does T^P_λ map $H^1_{prod}(\mathbb{R}^2)$ to $L^1(\mathbb{R}^2)$?

Further, there is a harder problem that also stands:

Does T^P_λ map $L\log L^k(\mathbb{R}^2)$ to $L^{1,\infty}(\mathbb{R}^2)$ for some $k \geq 1$?

A curiosity that also arises in this work is to develop an interpolation argument between $H^1_{rec}(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$. As far as we know such a result doesn't appear in the literature, but only interpolation arguments between $H^2_{prod}(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$ are known. If such a technique existed, then it would be easy to recover the general L^p theory.

Also, let us mention that we may attempt to obtain higher dimensional analogues of Corollary 2.3 as Corollary 2.2 remains valid in higher dimensions.

Finally, we highlight that the study of $\mathcal{H}^P$ is much more involved than that of T^P_λ, and thus every analogous question is open for this operator, including the validity of Theorem 2.4.

Chapter 3

Radial Multipliers in mixed norm spaces

3.1 A historical account.

Recall from the introduction that Fourier multipliers in $\mathbb{R}^n$, formally defined by

$$T_m f(x) := \int_{\mathbb{R}^n} m(\xi) \hat{f}(\xi) e^{2\pi i x \xi} d\xi, \tag{3.1.1}$$

have attracted great interest over the last century. To obtain a thorough account of the subject the reader is referred to [41]. Of particular interest is the operator associated to the multiplier $m(\xi) = \chi_{B(0,R)}(\xi)$, as it is intricately connected to the convergence of Fourier integrals. Note that

$$S_R f(x) = \int_{|\xi| \leq R} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

corresponds to the partial sums operator, and thus a bound such as $\|S_R f\|_p \lesssim \|f\|_p$ would imply the convergence of Fourier integrals in $L^p(\mathbb{R}^n)$. In 1971 C. Fefferman showed in a *tour de force* that the ball multiplier S_1 does not bound $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for any $p \neq 2$ when $n \geq 2$, [23] .

Remarkably enough, A. Córdoba [13] and G. Mockenhaupt [30] showed independently that the disc multiplier S_1 is nevertheless bounded in the polar mixed norm spaces $L^p_{rad} L^2_{ang}(\mathbb{R}^n)$ if and only if $\frac{2n}{n+1} < p < \frac{2n}{n-1}$. Let us point that also at the same time, J. L. Rubio de Francia showed the partial result concerned with the smoother Bochner-Riez means, [37]. The norm of these spaces is usually defined via polar coordinates in $\mathbb{R}^n$ by

$$\|f\|_{p,2} := \left(\int_0^\infty r^{n-1} \left(\int_{S^{n-1}} |f(r\theta)|^2 d\sigma(\theta) \right)^{\frac{p}{2}} dr \right)^{\frac{1}{p}}; \tag{3.1.2}$$

but they may also be expressed, by duality, by means of radial weights,

$$\|f\|_{p,2}^2 = \sup_{\omega \in L^q(\mathbb{R}^n)} \int_{\mathbb{R}^n} |f(x)|^2 \omega(x) dx, \tag{3.1.3}$$

where the supremum is taken over all radial functions $\omega \in L^q(\mathbb{R}^n)$ and q is the dual Hölder exponent of $\frac{p}{2}$, $\frac{1}{q} + \frac{2}{p} = 1$. A clear exposition of some basic properties of singular integrals in these mixed norm spaces may be found in [14]. The angular integrability imposed in these norms seems to be an effective way to defeat the geometrical difficulties arising in the classical L^p theory, namely the occurrence of Kakeya sets. The approach of A. Córdoba is based on the relationship between the Fourier transform and radial functions. Let us delve into this relation.

It is easy to check that the Fourier transform of a radial function is radial. However, a more involved subject is that of obtaining a formula to compute the Fourier transform of a radial function. To exploit the radial nature of a function f we approach the problem via polar coordinates,

$$\hat{f}(\xi) = \int_0^\infty r^{n-1} f(r) \int_{S^{n-1}} e^{-2\pi i x \xi} d\sigma(\theta)\, dr.$$

We thus face the need to compute the Fourier transform of the surface measure of the $n-$dimensional sphere,

$$\widehat{d\sigma}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(2\pi|\xi|),$$

where J_ν correspond to the Bessel functions, defined as in 2.0.2. Finally, we arrive at a suitable formula for the Fourier transform of a radial function f,

$$\hat{f}(\xi) = C|\xi|^{-\frac{n-2}{2}} \int_0^\infty r^{\frac{n}{2}} f(r) J_{\frac{n-2}{2}}(2\pi r|\xi|)\, dr. \tag{3.1.4}$$

Remark. Note that, by Plancherel theorem, $\|S_1\|_{2\to 2} \le \sup_\xi \left|\hat{\chi}_{B(0,1)}(\xi)\right|$. However a simple combination of the asymptotic obtained for Bessel functions as $r \to 0$ in (2.0.3) and (3.1.4) yield

$$\left|\hat{\chi}_{B(0,1)}(\xi)\right| \lesssim 1.$$

Although (3.1.4) is only valid for radial functions it may be complemented with the spherical harmonics decomposition of $L^2(\mathbb{S}^{n-1})$. By means of the Laplace operator on the sphere it is possible to obtain a orthonormal basis $\left\{Y_j^k\right\}_j^k$ of $L^2(\mathbb{S}^{n-1})$ called the spherical harmonics. That is, any function $f \in L^2(\mathbb{R}^n)$ can be expanded as a linear combination of spherical harmonics by

$$f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{d(k)} f_{k,j}(|x|)\, Y_k^j\left(\frac{x}{|x|}\right).$$

This expression already hints the advantage presented by $L^p_{rad}L^2_{ang}(\mathbb{R}^n)$ norms, as we may exploit the orthonormal nature of spherical harmonics. Clearly, given $f \in L^2(\mathbb{R}^n)$,

$$\|f\|_{p,2} = \left(\int_0^\infty r^{n-1}\left(\sum |f_{k,j}(r)|^2\right)^{\frac{p}{2}} dr\right)^{\frac{1}{p}}.$$

Further, spherical harmonics are also connected with the Fourier transform as shown in the following classical result.

Theorem 3.1. *Let $f \in L^2(\mathbb{R}^n)$, with $f(x) = f_0(|x|)\, Y_k^j\left(\frac{x}{|x|}\right)$ then*

$$\hat{f}(\xi) = 2\pi i^k Y_k^j\left(\frac{\xi}{|\xi|}\right)|\xi|^{-\frac{n+2k-2}{2}} \int_0^\infty f_0(r)\, J_{\frac{n+2k-2}{2}}(2\pi r|\xi|)\, dr. \tag{3.1.5}$$

Further, for a general $g \in L^2(\mathbb{R}^n)$, then

$$\hat{g}(\xi) = \sum_{k=0}^{\infty}\sum_{j=1}^{d(k)} 2\pi i^k Y_k^j\left(\frac{\xi}{|\xi|}\right)|\xi|^{-\frac{n+2k-2}{2}} \int_0^\infty g_j^k(r)\, J_{\frac{n+2k-2}{2}}(2\pi r|\xi|)\, dr, \tag{3.1.6}$$

where $\left\{g_j^k\right\}_j^k$ denotes the spherical harmonic expansion coefficients.

Once more, the reader is referred to [43] for a detailed introduction to the theory of spherical harmonics and a full proof of this result.

A. Córdoba realized that the boundedness of the disc multiplier follows from the following vector valued inequality:

Lemma 3.1 (A. Córdoba, [13])**.** *Let $\frac{2n}{n+1} < p < \frac{2n}{n-1}$ and let $\{g_\alpha\}_\alpha$ be a countable collection of functions in $\mathcal{S}(\mathbb{R})$. Then*

$$\int_0^\infty r^{(n-1)\left(1-\frac{p}{2}\right)}\left(\sum_\alpha |\mathcal{K}_\alpha g_\alpha(r)|^2\right)^{\frac{p}{2}} dr \lesssim \int_0^\infty r^{(n-1)\left(1-\frac{p}{2}\right)}\left(\sum_\alpha |g_\alpha(r)|^2\right)^{\frac{p}{2}} dr,$$

where the $\mathcal{K}_\alpha f(r) = \int_0^\infty f(t)\, k_\alpha(t,r)\, dt$ corresponds to an integral operator with kernel

$$\begin{aligned} k_\alpha(t,r) \quad = \quad & \frac{t^{\frac{1}{2}} J_\alpha(t)\, J_\alpha'(r)\, r^{\frac{1}{2}}}{t-r} - \frac{t^{\frac{1}{2}} J_\alpha'(t)\, J_\alpha(r)\, r^{\frac{1}{2}}}{t-r} \\ & \frac{t^{\frac{1}{2}} J_\alpha(t)\, J_\alpha'(r)\, r^{\frac{1}{2}}}{t+r} + \frac{t^{\frac{1}{2}} J_\alpha'(t)\, J_\alpha(r)\, r^{\frac{1}{2}}}{t+r}. \end{aligned}$$

The proof of this lemma is based on a careful analysis of the behaviour of Bessel functions, previously implemented by J. A. Barceló and A. Córdoba [2]. This study has been included in Appendix A.1 under Lemma A.1.

Later, A. Carbery *et al. [3, 4]* gave an elegant proof of the same result by means of the weighted theory and the universal Kakeya maximal function,

$$\mathcal{U}f(x) := \sup_{\substack{a,b>0 \\ \omega\in\mathbb{S}^{n-1}}} \frac{1}{a+b}\int_{-a}^{b} |f(x+t\omega)|\, dt.$$

Although this operator cannot be bounded in $L^p(\mathbb{R}^n)$ for any $1 \le p < \infty$, it behaves nicely if applied to radial functions.

Theorem 3.2 (A. Carbery, E. Hernández,F. Soria, [3])**.** *Let f be a radial function in $L^p(\mathbb{R}^n)$ with $p > n$, then $\|\mathcal{U}f\|_p \lesssim \|f\|_p$.*

The original proof appeared in [3] and a geometrical approach can be found in [14]. They then showed that (as suggested by E. Stein [38]) the L^2-weighted norm of the disc multiplier could be controlled by $\mathcal{U}$ for radial weights.

Theorem 3.3 (A. Carbery, E. Romera, F. Soria, [4]). *Given $\alpha > 1$, there exists a finite constant C_α such that for every radial weight ω, one has*

$$\int_{\mathbb{R}^n} |S_1 f(x)|^2 \omega(x)\, dx \leq C_\alpha \int_{\mathbb{R}^n} |f(x)|^2 \mathcal{U}_\alpha \omega(x)\, dx,$$

where $\mathcal{U}_\alpha \omega = \left(\mathcal{U}\left(|\omega|^\alpha\right)\right)^{\frac{1}{\alpha}}$.

It is then a simple matter of numerology to show that the ball multiplier is bounded in the range $\frac{2n}{n+1} < p < \frac{2n}{n-1}$.

Later, in 2012, J. Duoandikoetxea *et al.* [19] gave a characterization of a family of radial weights that allowed them to obtain the following estimate

Theorem 3.4 (J. Duoandikoetxea, A. Moyua, O. Oruetxebarria, [19]). *Let T_m be a Fourier multiplier defined as in (3.1.1) by*

$$(T_m f)\hat{}(\xi) := m(|\xi|)\, \hat{f}(\xi), \tag{3.1.7}$$

for all rapidly decreasing smooth functions f. Further, let m be a radial function of bounded variation satisfies the following hypothesis:

1. $\operatorname{Supp}(m) \subset [a,b] \subset \mathbb{R}^+$*, and m is differentiable in the interior (a,b).*
2. $\int_a^b |m'(x)|\, dx < \infty$.

Then T_m extends to a bounded operator in $L^p_{rad} L^2_{ang}(\mathbb{R}^n)$ if and only if $\frac{2n}{n+1} < p < \frac{2n}{n-1}$.

The first result we present in this thesis is a direct proof of this theorem in the spirit of A. Córdoba. In fact little more is needed to generalize A. Córdoba's proof for the disc multiplier [13] to a radial function of bounded variation. Further, combining this result with the boundedness properties of the universal maximal function we obtain another result in the spirit of Marcinkiewicz: Theorem 2.1.

Later we use Theorem 3.4 to show that a generalization of mixed norms in higher dimensions allows for more geometrical shapes. In $\mathbb{R}^{n+1}$, $n \geq 2$, we consider cylindrical coordinates (r, θ, z) where the first components (r, θ) correspond to the standard polar coordinates in $\mathbb{R}^n$; $0 < r < \infty$, $\theta \in S^{n-1}$, and $z \in \mathbb{R}$ denotes the zenithal coordinate. Note that will also use the notation (ρ, ϕ, ζ) to refer to the same coordinate system when working in the Fourier spectrum. In this coordinate system, the $L^p_{rad} L^2_{zen} L^2_{ang}\left(\mathbb{R}^{n+1}\right)$ norm is given by

$$\|f\|_{p,2,2} := \left(\int_0^\infty r^{n-1} \left(\int_{-\infty}^{\infty} \int_{S^{n-1}} |f(r,\theta,z)|^2\, d\theta dz \right)^{\frac{p}{2}} dr \right)^{\frac{1}{p}}. \tag{3.1.8}$$

We investigate the behaviour of the Fourier multiplier T_S, adapted to a solid S of revolution around the zenithal axis, formally defined by

$$T_S f(\xi, \zeta) := \int_{\mathbb{R}^n} \int_{\mathbb{R}} \chi_S(\xi,\zeta)\, \hat{f}(\xi,\zeta)\, e^{2\pi i (x\xi + z\zeta)} d\zeta d\xi, \tag{3.1.9}$$

where $\xi \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}$.

Theorem 3.5. *For all* $\frac{2n}{n+1} < p < \frac{2n}{n-1}$,

$$\|T_S f\|_{p,2,2} \lesssim \|f\|_{p,2,2}\,, \tag{3.1.10}$$

as long as $\frac{2n}{n+1} < p < \frac{2n}{n-1}$.

3.2 Marcinkiewicz type bounds for radial Fourier multipliers

In this section our aim is to prove Theorem 2.1. This result follows from combining slight modifications of Theorems 3.2 and 3.3 . However, in order to suitably modify these results we first give a direct proof of Theorem 3.4. This new (or rather old) approach brings to light the class of operators T^s defined in (2.0.4) that play an important role in the modifications to make.

3.2.1 Radial Fourier multipliers of bounded variation: revisited

We approach Theorem 3.4 in the spirit of A. Córdoba. Consider the operator T_m for a radial function $m : [0, \infty) \to \mathbb{R}$ formally defined by

$$T_m f(\xi) = \int_{\mathbb{R}^n} m(|\xi|)\, \hat{f}(\xi)\, e^{2\pi i x\xi} d\xi,$$

and combine this expression with the development of the Fourier transform in terms of its spherical harmonic expansion as in (3.1.6). T_m then becomes

$$T_m f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{d(k)} 2\pi i^k \int_{\mathbb{R}^n} e^{2\pi i x\xi} m(|\xi|)\, Y_k^j\left(\frac{\xi}{|\xi|}\right) |\xi|^{-\left(k+\frac{n-2}{2}\right)}$$
$$\int_0^\infty f_{k,j}(t)\, J_{k+\frac{n-2}{2}}(2\pi|\xi|t)\, t^{k+\frac{n-2}{2}} dt d\xi.$$

Exchanging the order of integration, the previous expression becomes

$$\sum_{k=0}^{\infty} \sum_{j=1}^{d(k)} 2\pi i^k \int_0^\infty f_{k,j}(t)\, t^{k+\frac{n-2}{2}}\, \hat{g}_t(x)\, dx,$$

where

$$g_t(\xi) = m(|\xi|)\, J_{k+\frac{n-2}{2}}(2\pi|\xi|t)\, |\xi|^{-\left(k+\frac{n-2}{2}\right)}\, Y_k^j\left(\frac{\xi}{|\xi|}\right).$$

Note that g_t has some radial structure, as $g_t(\xi) = g_0(|\xi|)\, Y_k^j(\xi)$ and thus we may invoke (3.1.5) to compute the Fourier transform of g_t. We then arrive to the expression

$$T_m f(r\theta) = \sum_{k=0}^{\infty} \sum_{j=1}^{d(k)} 4\pi^2 (-1)^k\, Y_k^j(\theta)\, T_m^{k,j} f(r),$$

with

$$T_m^{k,j} f(r) = \int_0^\infty f_{k,j}(t)\, t^{\frac{n+2k-1}{2}} r^{-\frac{n+2k-1}{2}} K_{k+\frac{n-2}{2}}(t,r)\, dt,$$

where

$$K_\alpha(t,r) = \sqrt{rt} \int_a^b m(s)\, J_\alpha(2\pi ts)\, J_\alpha(2\pi rs)\, sds.$$

In order to simplify the notation we will not keep track of the constants, that is we assume that

$$T_m f(r\theta) = \sum_{k=0}^{\infty} \sum_{j=1}^{d(k)} Y_k^j(\theta)\, T_m^{k,j} f(r) \tag{3.2.1}$$

with $T_m^{k,j}$ defined as before, but

$$K_\alpha(t,r) = \sqrt{rt} \int_a^b m(s)\, J_\alpha(ts)\, J_\alpha(rs)\, sds.$$

Let us take a closer look at the kernel of the operator K_α,

$$K_\alpha(t,r) = \sqrt{rt} \int_a^b m(s)\, J_\alpha(ts)\, J_\alpha(rs)\, sds. \tag{3.2.2}$$

In order to "simplify" the expression of this kernel we define an auxiliary function $\mathcal{U}_r(s) = \sqrt{rs} J_\alpha(rs)$. This expression together with Bessel's equation,

$$x^2 J_\alpha''(x) + x J_\alpha'(x) + \left(x^2 - \alpha^2\right) J_\alpha(x) = 0,$$

yields the following equality:

$$\frac{\partial}{\partial s} \left\{ \mathcal{U}_r(s)\, \mathcal{U}_t'(s) - \mathcal{U}_t(s)\, \mathcal{U}_r'(s) \right\} = \left(t^2 - r^2\right) \sqrt{tr} J_\alpha(rs)\, J_\alpha(ts)\, s.$$

Therefore, after an integration by parts in (3.2.2), we obtain

$$\begin{aligned} K_\alpha(t,r) &= \left[m(s) \frac{1}{t^2 - r^2} \left\{ \mathcal{U}_r(s)\, \mathcal{U}_t'(s) - \mathcal{U}_t(s)\, \mathcal{U}_r'(s) \right\} \right]_a^b \\ &\quad - \int_a^b m'(s) \frac{1}{t^2 - r^2} \left\{ \mathcal{U}_r(s)\, \mathcal{U}_t'(s) - \mathcal{U}_t(s)\, \mathcal{U}_r'(s) \right\} ds. \end{aligned}$$

Hence, we may express the the operator T_m in the following way:

$$\begin{aligned} T_m f(r\theta) &= \sum_{k,j} Y_k^j(\theta) \int_0^\infty f_{k,j}(t)\, t^{\frac{n+2k-1}{2}} r^{-\frac{n+2k-1}{2}} \cdot \\ &\cdot \left(m(b)\, k_\alpha(r,t,b) - m(a)\, k_\alpha(r,t,a) - \int_a^b m'(s)\, k_\alpha(r,t,s)\, ds \right) dt, \end{aligned} \tag{3.2.3}$$

where $\alpha = k + \frac{n-2}{2}$ and $k_\alpha(t,r,s) = \frac{1}{t^2 - r^2} \left\{ \mathcal{U}_r(s)\, \mathcal{U}_t'(s) - \mathcal{U}_t(s)\, \mathcal{U}_r'(s) \right\}$. A simple expansion of k_α reveals the underlying singularities of the operator K_α;

$$\begin{aligned} k_\alpha(t,r,s) &= \left(s \frac{\sqrt{t} J_\alpha'(ts)\, J_\alpha(rs) \sqrt{r}}{2(t-r)} + s \frac{\sqrt{t} J_\alpha'(ts)\, J_\alpha(rs) \sqrt{r}}{2(t+r)} \right. \\ &\quad \left. + s \frac{\sqrt{t} J_\alpha(ts)\, J_\alpha'(rs) \sqrt{r}}{2(r-t)} + s \frac{\sqrt{t} J_\alpha'(ts)\, J_\alpha(rs) \sqrt{r}}{2(t+r)} \right). \end{aligned} \tag{3.2.4}$$

A thorough study of the kernel $k_\alpha(r,t,1)$ was carried out in [13] using the decay properties of Bessel functions (Appendix A.1, Lemma A.1) to produce Lemma 3.1. Although nothing really new has been done, we have brought to light a more general family of operators underlying the disc multiplier, that is the family of operators T^s defined as

$$T^s f(r\theta) = \sum_{k,j} Y_k^j(\theta) \int_0^\infty f_{k,j}(t)\, t^{\frac{n+2k-1}{2}} r^{-\frac{n+2k-1}{2}} k_\alpha(r,t,s)\, dt. \tag{3.2.5}$$

Let us highlight that T^1 corresponds to the disc multiplier $T_{\chi_{B(0,1)}}$.

Indeed, a bound of the operator T^s uniform in s implies a bound on T_m. Note that

$$\|T_m f\|_{p,2} \lesssim |m(b)| \left\|T^b f\right\|_{p,2} + |m(a)| \|T^a f\|_{p,2} + \int_a^b |m'(s)| \|T^s f\|_{p,2}\, ds, \tag{3.2.6}$$

and thus Theorem 3.4 follows from the uniformity of the bound in Proposition 2.1, that is

$$\|T_m f\|_{p,2} \le C \left(\sup_{s\in[a,b]} |m(s)| + \int_a^b |m'(s)|\, ds \right) \|f\|_{p,2}.$$

Proof of Proposition 2.1. In order to simplify the expression of the kernel of T^s we will just write one of the four summands of k_α, that is

$$T^s f(r\theta) \sim \sum_{k,j} Y_k^j(\theta) \int_0^\infty f_{k,j}(t)\, t^{\frac{n+2k-1}{2}} r^{-\frac{n+2k-1}{2}} s \frac{\sqrt{t} J'_\alpha(ts) J_\alpha(rs) \sqrt{r}}{2(t-r)} dt, \tag{3.2.7}$$

for any fixed $s \in (0,\infty)$. The orthonormality in $L^2\left(\mathbb{S}^{n-1}\right)$ of spherical harmonics can now be used in our advantage to compute the $L^p_{rad}L^2_{ang}$ norm of T^s. Indeed, $\|T^s f\|_{p,2,}$ is up to the notation reduction equal to

$$\left(\int_0^\infty r^{n-1} \left\{ \sum_{k,j} \left| \int_0^\infty f_{k,j}(t)\, t^{\frac{n+2k-1}{2}} r^{-\frac{n+2k-1}{2}} s \frac{\sqrt{t} J'_\alpha(ts) J_\alpha(rs) \sqrt{r}}{2(t-r)} dt \right|^2 \right\}^{\frac{p}{2}} dr \right)^{\frac{1}{p}}.$$

Two simple changes of variables, $t' = st$ and $r' = sr$, yield

$$s^{-\frac{n}{p}} \left(\int_0^\infty r^{n-1} \left\{ \sum_{k,j} \left| \int_0^\infty f_{k,j}\left(\frac{t}{s}\right) t^{\frac{n+2k-1}{2}} r^{-\frac{n+2k-1}{2}} \frac{\sqrt{t} J'_\alpha(t) J_\alpha(r) \sqrt{r}}{2(t-r)} dt \right|^2 \right\}^{\frac{p}{2}} dr \right)^{\frac{1}{p}}. \tag{3.2.8}$$

Therefore, applying Lemma 3.1 we may bound this expression by

$$C_{p,n} s^{-\frac{n}{p}} \left(\int_0^\infty r^{n-1} \left\{ \sum_{k,j} \left| f_{k,j}\left(\frac{r}{s}\right) \right|^2 \right\}^{\frac{p}{2}} dr \right)^{\frac{1}{p}},$$

for every $\frac{2n}{n+1} < p < \frac{2n}{n-1}$. Finally, a last change of variables produces the estimate

$$\|T^s f\|_{p,2} \leq C \|f\|_{p,2},$$

where C is uniform on s. □

3.2.2 A mixed norm Littlewood Paley theory

The natural next step after showing Theorem 3.4 is to deduce a Marcinkiewicz type theorem. This section aims to give a proof of Theorem 2.1. The obvious approach to this problem is to use Littlewood-Paley techniques in this mixed norm setting.

Let T be a radial Fourier multiplier operator and denote by T_j the restriction of T to the dyadic interval $I_j = \left[2^j, 2^{j+1}\right]$. The usual Littlewood-Paley techniques yield

$$\|Tf\|_{p,2} \approx \left\| \left(\sum_j |T_j f|^2 \right)^{\frac{1}{2}} \right\|_{p,2} \approx \left\| \left(\sum_j |T_j S_j f|^2 \right)^{\frac{1}{2}} \right\|_{p,2}, \tag{3.2.9}$$

where S_j is the classical Littlewood-Paley operator

$$(S_j f)\hat{}(\xi) := \chi_{[2^j,2^{j+1}]}(\xi) \hat{f}(\xi). \tag{3.2.10}$$

The proof of Theorem 2.1 is then a consequence of

$$\left\| \left(\sum_j |T_j S_j f|^2 \right)^{\frac{1}{2}} \right\|_{p,2} \lesssim \left\| \left(\sum_j |S_j f|^2 \right)^{\frac{1}{2}} \right\|_{p,2}, \tag{3.2.11}$$

since as usual $\left\| \left(\sum_j |S_j f|^2 \right)^{\frac{1}{2}} \right\|_{p,2} \sim \|f\|_{p,2}$. In order to show this inequality we need gain a control of T^s by means of the weighted theory and the universal Kakeya maximal function, in the spirit of Theorem 3.3. This control is detailed in the following Lemma.

Lemma 3.2. *Let T^s be the generalized disc multiplier defined as previously (2.0.4). For every $\beta > 1$ and every radial weight g there exists a finite constant C_β so that for every rapidly decreasing function f,*

$$\int_{\mathbb{R}^n} |T^s f(x)|^2 g(x)\, dx \leq C_\beta \int_{\mathbb{R}^n} |f(x)|^2 \mathcal{M}_\beta g(x)\, dx, \tag{3.2.12}$$

where the constant C_β is uniform in s. Here $\mathcal{M}$ denotes the universal Kakeya maximal function and $\mathcal{M}_\beta g = \left(\mathcal{M}\left(|g|^\beta\right)\right)^{\frac{1}{\beta}}$.

Proof. Recall the expansion of T^s in spherical harmonics and develop the left hand side of (3.2.12) in spherical coordinates;

$$\int_0^\infty r^{n-1} \int_{\mathbb{S}^{n-1}} g(r\theta) \left| \sum_{k,j} Y_k^j(\theta) \int_0^\infty f_{k,j}(t)\, t^{\frac{n+2k-1}{2}} r^{-\frac{n+2k-1}{2}} k_\alpha(r,t,s)\, dt \right|^2 dr. \tag{3.2.13}$$

Again, as in (3.2.8), the changes of variables $t' = st$ and $r' = sr$ show that

$$\int_{\mathbb{R}^n} |T^s f(x)|^2 g(x)\, dx = s^{-n} \int_{\mathbb{R}^n} \left| T_{\chi_{B(0,1)}} f\left(\frac{x}{s}\right)\right|^2 g\left(\frac{x}{s}\right) dx, \tag{3.2.14}$$

since $T^1 = T_{\chi_{B(0,1)}}$. We can therefore use Theorem 3.3 to transfer the complexity of the disc multiplier into that of the universal Kakeya maximal function

$$\int_{\mathbb{R}^n} |T_0 f(x)|^2 g(x)\, dx \lesssim \int_{\mathbb{R}^n} |f(x)|^2 \mathcal{M}_\alpha g(x)\, dx.$$

Once more, it is an easy task to undo the change of variables in the radial variable to obtain

$$\int_{\mathbb{R}^n} |T^s f(x)|^2 g(x)\, dx \le C \int_0^\infty |f(x)|^2 \mathcal{M}_\alpha g(x)\, dx,$$

with constant C not depending on s. □

We are now ready to finish the proof of the boundedness of Marcinkiewicz type Fourier multipliers.

Proof of Theorem 2.1. Following the ideas already developed, it is enough to show that

$$\left\| \left(\sum_j |T_j S_j f|^2 \right)^{\frac{1}{2}} \right\|_{p,2} \lesssim \left\| \left(\sum_j |S_j f|^2 \right)^{\frac{1}{2}} \right\|_{p,2}.$$

However, using the dual definition of $L^p_{rad} L^2_{ang}(\mathbb{R}^n)$ spaces 3.1.3,

$$\left\| \left(\sum_j |T_j S_j f|^2 \right)^{\frac{1}{2}} \right\|_{p,2}^2 = \sup_{\|g\|_q = 1} \sum_j \int_{\mathbb{R}^n} |T_j S_j f(r\theta)|^2 g(x)\, dx,$$

where the supremum is taken over all radial functions in $L^q(\mathbb{R}^n)$, with $\frac{1}{q} + \frac{2}{p} = 1$.

Therefore, following the development of T_j as detailed in the proof of Theorem 3.4 and using Lemma 3.2 we may bound this expression by

$$\sup_{\|g\|_q = 1} (2\|m\|_\infty + C) \sum_j \int_{\mathbb{R}^n} |S_j f(r\theta)|^2 \mathcal{M}_\beta g(x)\, dx, \tag{3.2.15}$$

for some $\beta > 0$, where the constant C is detailed in (2.0.7)and corresponds to the uniform bound $\int_{I_j} |m'(s)|\, ds < C$ for every dyadic interval I_j. Then Hölder's inequality with $\frac{1}{q} + \frac{2}{p} = 1$ and $\frac{2n}{n+1} < p < \frac{2n}{n-1}$ yields

$$\left\| \left(\sum_j |T_j S_j f|^2 \right)^{\frac{1}{2}} \right\|_{p,2} \lesssim \left(\int_0^\infty r^{n-1} (\mathcal{M}_\alpha g(r))^q\, dr \right)^{\frac{1}{q}} \left\| \left(\sum |S_j f|^2 \right)^{\frac{1}{2}} \right\|_{p,2}^2.$$

We finish the proof making use of the boundedness of the universal Kakeya function in $L^q(\mathbb{R}^n)$ for every $q > n$, Theorem 3.2. □

3.3 Multiplier associated to a solid of revolution

We now aim to bound the $L^p_{rad}L^2_{ang}L^2_{zen}\left(\mathbb{R}^{n+1}\right)$ norm of the operator T_S adapted to the solid of revolution S, as presented in Theorem 3.5. First, we parametrize the solid of revolution S by its generating function g;

$$S := \{(x, z) \in \mathbb{R}^n \times \mathbb{R},\ |x| \leq g(z)\},$$

where g is taken to be a continuous functions with compact support say in $[-a, a]$, for some $a > 0$.

We first need to express $f_z(x)$ in its spherical harmonics expansion,

$$f(x, z) := \sum_{k=0}^{\infty}\sum_{j=1}^{d(k)} a_{k,j}(|x|, z)\, Y_k^j\left(\frac{x}{|x|}\right), \tag{3.3.1}$$

and compute its Fourier transform in the first n parameters using the classical formula (3.1.6). We then obtain the following expression of $\hat{f}(\xi, \zeta)$:

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-2\pi i z\zeta}\sum_{k,j} 2\pi i^k Y_k^j\left(\frac{\xi}{|\xi|}\right)|\xi|^{-\left(k+\frac{n-2}{2}\right)}\int_0^\infty a_{k,j}(t, z)\, J_{k+\frac{n-2}{2}}(t|\xi|)\, t^{k+\frac{n}{2}}\, dt \\ = \sum_{k,j} 2\pi i^k Y_k^j\left(\frac{\xi}{|\xi|}\right)|\xi|^{-\left(k+\frac{n-2}{2}\right)}\int_0^\infty \left(a^t_{k,j}\right)\hat{}(\zeta)\, J_{k+\frac{n-2}{2}}(t|\xi|)\, t^{k+\frac{n}{2}}\, dt.\end{aligned}$$

In this last equality, $\left(a^t_{k,j}\right)\hat{}(\zeta)$ refers to the Fourier transform of $a_{k,j}(t, z)$ in the second variable. As usual we replace this expression in (3.1.9),

$$\begin{aligned}T_S f(x, z) = \sum_{k,j} 2\pi i^k \int_{-\infty}^{\infty} e^{2\pi i z\zeta}\chi_{[-a,a]}(\zeta) \\ \int_{\mathbb{R}^n} e^{2\pi i x\xi} Y_k^j\left(\frac{\xi}{|\xi|}\right)\chi_{[0,g(\zeta)]}(|\xi|)\,|\xi|^{-\left(k+\frac{n-2}{2}\right)}\int_0^\infty \left(a^t_{k,j}\right)\hat{}(\zeta)\, J_{k+\frac{n-2}{2}}(t|\xi|)\, t^{k+\frac{n}{2}}\, dt d\xi d\zeta.\end{aligned}$$

Once again, the second line of this expression is the Fourier transform of the product of a radial function and a spherical harmonic. Therefore, by (3.1.5), the operator T_S becomes

$$\begin{aligned}T_S f(x, z) = \sum_{k,j}\left(2\pi i^k\right)^2 Y_k^j\left(\frac{x}{|x|}\right)\int_{-\infty}^{\infty} e^{2\pi i z\zeta}\chi_{[-a,a]}(\zeta) \\ \int_0^\infty \left(a^t_{k,j}\right)\hat{}(\zeta)\, t^{k+\frac{n-1}{2}}\,|x|^{-\left(k+\frac{n-1}{2}\right)} \\ \sqrt{t|x|}\int_0^\infty \chi_{[0,g(\zeta)]}(s)\, J_{k+\frac{n-2}{2}}(2\pi st)\, J_{k+\frac{n-2}{2}}(2\pi s|x|)\, s ds dt d\zeta.\end{aligned}$$

Using the same trick as in (3.2.2) and denoting as usual $r = |x|$ and $\theta = \frac{x}{|x|}$, we may rewrite this expression up to the terms $2\pi i^k$ as

$$\begin{aligned}T_S f(r\theta, z) = \sum_{k,j} Y_k^j(\theta)\int_{-\infty}^{\infty} e^{2\pi i z\zeta}\chi_{[-a,a]}(\zeta) \\ \int_0^\infty \left(a^t_{k,j}\right)\hat{}(\zeta)\, t^{k+\frac{n-1}{2}} r^{-\left(k+\frac{n-1}{2}\right)} k_{k+\frac{n-2}{2}}(r, t, g(\zeta))\, dt d\zeta,\end{aligned}$$

where the kernels $k_\alpha(r,t,s)$ correspond to those detailed in (2.0.5). To further simplify the notation, we denote by $\left(a_{k,j}^{t,r}\right)\hat{}(\zeta) = \left(a_{k,j}^{t}\right)\hat{}(\zeta)\, t^{k+\frac{n-1}{2}} r^{-\left(k+\frac{n-1}{2}\right)}$ and obtain

$$T_S f(r\theta, z) = \sum_{k,j} Y_k^j(\theta) \int_{-\infty}^{\infty} e^{2\pi i z\zeta} \chi_{[-a,a]}(\zeta)$$
$$\int_0^\infty \left(a_{k,j}^{t,r}\right)\hat{}(\zeta)\, k_{k+\frac{n-2}{2}}(r,t,g(\zeta))\, dt d\zeta. \quad (3.3.2)$$

As we are interested in the $L^p_{rad} L^2_{ang} L^2_{zen}\left(\mathbb{R}^{n+1}\right)$ norm of the operator T_S let us inquire a bit more about this space of functions. First define $I(f,\omega)$ for $g : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and $\omega : \mathbb{R} \to \mathbb{R}$ as follows

$$\begin{aligned} I(g,w) : &= \int_{\mathbb{R}^n} \omega(|x|) \|f(x,z)\|^2_{L^2_z(\mathbb{R})} dx \\ &= \int_0^\infty r^{n-1}\omega(r) \int_{\mathbb{S}^{n-1}} \|g(r\theta,z)\|^2_{L^2_z(\mathbb{R})} d\sigma(\theta)\, dr, \\ &= \int_0^\infty r^{n-1}\omega(r) \left(\left\| \|gf(r\theta,z)\|_{L^2_z(\mathbb{R})} \right\|_{L^2(\mathbb{S}^{n-1})} \right)^2 dr. \quad (3.3.3) \end{aligned}$$

We have thus obtained a dual characterization of the cylindrical mixed norm space,

$$\|f\|^2_{p,2,2} = \sup_{\|\omega\|_{L^q(\mathbb{R}^n)} \le 1} I(f,\omega), \quad (3.3.4)$$

for where q corresponds to the Hölder dual exponent of $\frac{p}{2}$. Notice that we are identifying the function ω to its radial analogue in higher dimension$\omega_{rad}(x) = \omega(|x|)$.

On another hand, using the spherical harmonics expansion (3.3.1) on the first n variables of $\mathbb{R}^{n+1}$,

$$\|f(r\theta,z)\|^p_{p,2,2} = \int_0^\infty r^{n-1} \left(\int_{\mathbb{S}^{n-1}} \int_{-\infty}^{\infty} \left| \sum_{k,j} a_{k,j}(r,z)\, Y_k^j(\theta) \right|^2 dz d\sigma(\theta) \right)^{\frac{p}{2}} dr.$$

Further, Plancherel's theorem in the last variable z yields

$$\|f(r\theta,z)\|^p_{p,2,2} = \int_0^\infty r^{n-1} \left(\int_{\mathbb{S}^{n-1}} \int_{-\infty}^{\infty} \left| \sum_{k,j} \left(a_{k,j}^r\right)\hat{}(\zeta)\, Y_k^j(\theta) \right|^2 dz d\sigma(\theta) \right)^{\frac{p}{2}} dr,$$

where again $\left(a_{k,j}^r\right)\hat{}(\zeta)$ corresponds to the Fourier transform of $a_{k,j}(r,z)$in the second variable. We may also switch the angular and zenithal order of integration and use the fact that spherical harmonics form an orthonormal basis of $\mathbb{S}^{n-1}$, that is

$$\left\| \sum_{k,j} Y_k^j(\theta) \left(a_{k,j}^r\right)\hat{}(\zeta) \right\|_{L^2(\mathbb{S}^{n-1})} = \left(\sum \left| \left(a_{k,j}^r\right)\hat{}(\zeta) \right|^2 \right)^{\frac{1}{2}}.$$

We may thus also obtain a spherical harmonics characterization of the cylindrical mixed norm space

$$\|f\|_{L^{p,2,2}} = \left\{ \int_0^\infty r^{n-1} \left(\int_{-\infty}^{\infty} \sum_{k,j} \left| \left(a_{k,j}^r\right)\hat{}(\zeta) \right|^2 d\zeta \right)^{\frac{p}{2}} dr \right\}^{\frac{1}{p}}.$$

The statement of Theorem 2.2 is thus equivalent to showing that

$$\sup_{\|\omega\|_{L^q(\mathbb{R})} \leq 1} I(Tf,\omega) \lesssim \left\{ \int_0^\infty r^{n-1} \left(\int_{-\infty}^{\infty} \sum_{k,j} \left| \left(a_{k,j}^r\right)\hat{}(\zeta) \right|^2 d\zeta \right)^{\frac{p}{2}} dr \right\}^{\frac{2}{p}},$$

where $\frac{1}{q} + \frac{2}{p} = 1$.

Let us slowly compute the $I(f,\omega)$ by means of the expansion (3.3.2) of T_S. First of all, note that we can use Plancherel's Theorem in the L_z^2 norm to obtain

$$\|Tf(r\theta,z)\|_{L_z^2(\mathbb{R})} = \left\|\tilde{T}f(r\theta,z)\right\|_{L_z^2(\mathbb{R})},$$

where $\tilde{T}$

$$\tilde{T}f(r\theta,z) = \sum_{k,j} Y_k^j(\theta)\,\chi_{[-a,a]}(z) \int_0^\infty \left(a_{k,j}^{t,r}\right)\hat{}(z)\, k_{k+\frac{n-2}{2}}(r,t,g(z))\,dt.$$

Then, we can also swap the order between the zenithal and angular integration in (3.3.3), so that

$$I(f,\omega) = \int_0^\infty r^{n-1}\omega(r) \left(\left\| \left\| \tilde{T}f(r\theta,z) \right\|_{L^2\left(\mathbb{S}^{n-1}\right)} \right\|_{L_z^2(\mathbb{R})} \right)^2 dr.$$

But we can once more take into account that $\left\{Y_k^j\right\}_k^j$ forms an orthonormal basis of $L^2\left(\mathbb{S}^{n-1}\right)$. Hence we may express $I(f,\omega)$

$$\int_{-\infty}^{\infty} \chi_{[-a,a]}(z) \sum_{k,j} \int_0^\infty r^{n-1}\omega(r) \left| \int_0^\infty \left(a_{k,j}^{t,r}\right)\hat{}(z)\, k_{k+\frac{n-2}{2}}(r,t,g(z))\,dt \right|^2 dr dz.$$

Let us expand the kernel in the last equation and do the changes of variables $t' = g(z)\,t$ and $r' = g(z)\,r$ and decode the expression in terms of the auxiliary operator T^s:

$$\begin{aligned}
&\int_{-\infty}^{\infty} g(z)^{5-(n-1)} \chi_{[-a,a]}(z) \\
&\qquad \sum_{k,j} \int_0^\infty r^{n-1} w\left(\frac{r}{g(z)}\right) \left| \int_0^\infty \left(a_{k,j}^{\frac{t}{g(z)},\frac{r}{g(z)}}\right)\hat{}(z)\, k_{k+\frac{n-2}{2}}(r,t,1)\,dt \right|^2 dr dz,
\end{aligned}$$

and we can control the inner operator by means of the universal Kakeya maximal function as in Lemma 3.2 by

$$\int_0^\infty r^{n-1} \mathcal{M}_\beta \omega\left(\frac{r}{g(z)}\right) \left| \left(a_{k,j}^{\frac{r}{g(z)},\frac{r}{g(z)}}\right)\hat{}(z) \right|^2 dr.$$

We thus have the following inequality

$$I(f,w) \lesssim \int_{-\infty}^{\infty} g(z)^{5-(n-1)} \chi_{[-a,a]}(z) \sum_{k,j} \int_0^{\infty} r^{n-1} \mathcal{M}_s \omega\left(\frac{r}{g(z)}\right) \left|\left(a_{k,j}^{\frac{r}{g(z)},\frac{r}{g(z)}}\right)\hat{}(z)\right|^2 drdz.$$

Let us highlight that

$$\left(a_{k,j}^{t,r}\right)\hat{}(z) = \int_{-\infty}^{\infty} e^{2\pi i z\zeta} a_{k,j}(t,\zeta) r^{-\left(k+\frac{n-1}{2}\right)} t^{k+\frac{n-1}{2}} d\zeta,$$

hence

$$\begin{aligned}\left(a_{k,j}^{\frac{r}{g(z)},\frac{r}{g(z)}}\right)\hat{}(z) &= \int_{-\infty}^{\infty} e^{2\pi i z\zeta} a_{k,j}\left(\frac{r}{g(z)},\zeta\right)\left(\frac{r}{g(z)}\right)^{-\left(k+\frac{n-1}{2}\right)}\left(\frac{r}{g(z)}\right)^{k+\frac{n-1}{2}} \\ &= \left(a_{k,j}^{\frac{r}{g(z)}}\right)\hat{}(z).\end{aligned}$$

We can finally bound $I(f,\omega)$ by

$$\begin{aligned} &\int_{-\infty}^{\infty} \chi_{[-a,a]}(z)\, g(z)^{5-(n-1)} \sum_{k,j} \int_0^{\infty} r^{n-1} \mathcal{M}_s \omega\left(\frac{r}{g(z)}\right) \left|\left(a_{k,j}^{\frac{r}{g(z)}}\right)\hat{}(z)\right|^2 drdz \\ = &\int_{-\infty}^{\infty} g(z)^4 \int_0^{\infty} r^{n-1} \mathcal{M}_s \omega(r) \sum_{k,j} \left|\left(a_{k,j}^{r}\right)\hat{}(z)\right|^2 drdz \\ \leq &\ \|g\|_{\infty}^4 \int_0^{\infty} r^{n-1} \mathcal{M}_s \omega(r) \int_{-\infty}^{\infty} \sum_{k,j} \left|\left(a_{k,j}^{r}\right)\hat{}(z)\right|^2 dzdr. \end{aligned}$$

We have reverted a change of variables in the radial integration $r' = g(z)^{-1} r$ and disposed of $\chi_{[-a,a]}(z)$ by taking the supremum. We need only to use Holder's inequality with $\frac{2}{p} + \frac{1}{q} = 1$, to obtain

$$I(f,\omega) \lesssim \left(\int_0^{\infty} r^{n-1} \left\{\int_{-\infty}^{\infty} \sum_{k,j} \left|\left(a_{k,j}^{r}\right)\hat{}(z)\right|^2 dz\right\}^{\frac{p}{2}} dr\right)^{\frac{2}{p}} \times \left(\int_0^{\infty} r^{n-1} \left|\mathcal{M}_s \omega(r)\right|^q dr\right)^{\frac{1}{q}}.$$

This finishes the proof as $\mathcal{M}_s$ acting on radial functions is a bounded operator on $L^{\left(\frac{p}{2}\right)'}(\mathbb{R}^n)$ for $\frac{2n}{n+1} < p < \frac{2n}{n-1}$. Thus, taking the supremum over all $\omega \in L^q(\mathbb{R}^n)$ with $\|\omega\|_q \lesssim 1$, following

$$\|Tf\|_{L^{p,2,2}}^2 \leq C_g \|f\|_{L^{p,2,2}}^2,$$

where the constant C depends on the supremum$\|g\|_{\infty}$.

Chapter 4

Restriction Theorem

In this chapter we center our attention around the well known restriction conjecture. Recall that for any $f \in L^2(\mathbb{R}^n)$ its Fourier transform $\mathcal{F}f$ is at best a square integrable function. That is $\mathcal{F}f$ is only well defined up to sets of measure zero and may have severe singularities in these sets. However, the Fourier transform has some extra structure when acting on spaces $L^p(\mathbb{R}^n)$ for p close to 1. This phenomena is apparent when we look at the action of $\mathcal{F}$ in the space of integrable functions. It is an easy consequence of the Lebesgue dominated convergence theorem to see that $\mathcal{F} : L^1(\mathbb{R}^n) \to \mathcal{C}_0(\mathbb{R}^n)$, where $\mathcal{C}_0$ is the space of all continuous functions with decay at infinity. Therefore, if $f \in L^1(\mathbb{R}^n)$, $\mathcal{F}f$ cannot be singular on any set of measure zero. This observation led, in the mid 1960s, E. M. Stein to realize that the Fourier transform of a function $f \in L^p(\mathbb{R}^n)$ cannot be too singular in spheres. That is, if we consider the operator $\mathcal{R}f = \hat{f}|_{\mathbb{S}^{n-1}}$we can make sense of $\|\mathcal{R}f\|_{L^q(\mathbb{S}^{n-1})}$ for some q as long as $f \in L^p(\mathbb{R}^n)$ with p close enough to 1. The precise range of admissible pairs (p,q) specified in (2.0.10) is derived from *Knapp's example* (see [47]). I was not until much later that P. Tomas [47] and E. M. Stein [40] determined that, in the special case where $q = 2$,

$$\|\mathcal{R}f\|_{L^2(\mathbb{S}^{n-1})} \leq C \|f\|_{L^p(\mathbb{R}^n)}$$

holds for the range $1 \leq p \leq \frac{2(n+1)}{n+3}$. This answer is, however, unsatisfactory as it does not cover the full conjectured range of admissible p.

In his PhD Thesis [48], L. Vega showed that, if in $\mathbb{R}^n$ we establish polar coordinates $L^p_{rad}L^2_{ang}(\mathbb{R}^n)$, the restriction conjecture with $q = 2$ holds true in this mixed norm setting. In Theorem 2.3 we push this result a bit further and show that, if in $\mathbb{R}^{n+1}$ we consider cylindrical coordinates $L^p_{rad}L^2_{ang}L^2_{zen}\left(\mathbb{R}^{n+1}\right)$ (as defined in (2.0.9)), we may replace spheres by C^1 compact surfaces of revolution Γ.

More precisely, let g be a continuous positive function supported on a compact interval I of the real line that is almost everywhere differentiable, and consider the surface of revolution in $\mathbb{R}^{n+1}$ given by

$$\Gamma := \left\{(g(z), \theta, z) \in \mathbb{R}^{n+1},\ \theta \in S^{n-1}, -\infty < z < \infty\right\}, \tag{4.0.1}$$

then

$$\left\|\hat{f}\right\|_{L^2(\Gamma)} \leq C_p \|f\|_{L^{p,2,2}(\mathbb{R}^{n+1})}$$

holds for all $1 \leq p < \frac{2n}{n+1}$.

It is important to note that the restriction conjecture can be stated by duality as an *extension conjecture*. In this case, the statement of Theorem 2.3, in the introduction, asserts that for any rapidly decreasing function f,

$$\left\| \widehat{fd\Gamma} \right\|_{L^{q,2,2}(\mathbb{R}^{n+1})} \leq C_q \left\| f \right\|_{L^2(\Gamma)} \tag{4.0.2}$$

holds as long as $q > \frac{2n}{n-1}$. This is the statement we prove in Theorem 2.3. As in the previous results concerned with mixed norm estimates, we first need to expand this inequality in terms of spherical harmonics.

We first focus on the expansion of $\widehat{f\Gamma}$. Note that since Γ is generated by the curve g,

$$\begin{aligned} d\Gamma &= g\left(z\right)^{n-1} \sqrt{1 + \left(g'\left(z\right)\right)^2} dz d\theta \\ &= G_1\left(z\right) dz d\theta. \end{aligned}$$

Therefore

$$\widehat{fd\Gamma}\left(\rho, \phi, \zeta\right) = \int_{-\infty}^{\infty} \int_{S^{n-1}} G_1\left(z\right) f\left(g\left(z\right), \theta, z\right) e^{-iz\zeta} e^{-i(\rho g(z))\theta \cdot \phi} d\theta dz. \tag{4.0.3}$$

Next we use the spherical harmonic expansion (3.1.6) to develop f

$$f\left(g\left(z\right), \theta, z\right) = \sum_{k,j} a_{k,j}\left(z\right) Y_k^j\left(\theta\right).$$

We then obtain

$$\begin{aligned} \widehat{fd\Gamma}\left(\rho, \phi, \zeta\right) = \sum_{k,j} 2\pi i^k Y_k^j\left(\phi\right) \rho^{-\frac{n-2}{2}} \int_{-\infty}^{\infty} g\left(z\right)^{\frac{n}{2}} \left(1 + \left(g'\left(z\right)\right)^2\right)^{\frac{1}{2}} \cdot \\ \cdot a_{k,j}\left(z\right) J_{k+\frac{n-2}{2}}\left(\rho g\left(z\right)\right) e^{-iz\zeta} dz, \end{aligned}$$

where J_ν denotes, as usual, the Bessel's function of order ν. For notation purposes, we introduce an auxiliary function $G_2\left(z\right) := g\left(z\right)^{\frac{n}{2}} \left(1 + \left(g'\left(z\right)\right)^2\right)^{\frac{1}{2}}$ and the Fourier transform $\widehat{fd\Gamma}$ is then expressed by

$$\sum_{k,j} 2\pi i^k Y_k^j\left(\phi\right) \rho^{-\frac{n-2}{2}} \int_{-\infty}^{\infty} G_2\left(z\right) a_{k,j}\left(z\right) J_{k+\frac{n-2}{2}}\left(\rho g\left(z\right)\right) e^{-iz\zeta} dz. \tag{4.0.4}$$

Once more, we exploit the structure of $L^2_{ang}\left(\mathbb{S}^{n-1}\right)$ by means of the orthogonal nature of the elements of the basis $\left\{Y_k^j\right\}$. Further, Plancherel's Theorem in the z-variable yields that the mixed norm $\left\|\widehat{fd\Gamma}\right\|^q_{L^{q,2,2}}$ is up to a constant equal to

$$\int_0^{\infty} \rho^{-q\frac{n-2}{2}+n-1} \left(\sum_{k,j} \int_{-\infty}^{\infty} \left|g\left(\zeta\right)\right|^n \left|1 + \left(g'\left(\zeta\right)\right)^2\right| \left|a_{k,j}\left(\zeta\right)\right|^2 \left|J_{\nu_k}\left(\rho g\left(\zeta\right)\right)\right|^2 d\zeta \right)^{\frac{q}{2}} d\rho, \tag{4.0.5}$$

where $\nu_k = k + \frac{n-2}{2}$. On the other hand we have

$$\begin{aligned}\int_\Gamma |f|^2 &= \int_{-\infty}^{\infty} \int_{S^{n-1}} \left| \sum_{j,k} a_{k,j}(z)\, Y_k^j(\theta) \right|^2 g(z)^{n-1} \sqrt{1+g'(z)^2}\, d\theta dz \\ &= \sum_{j,k} \int_{-\infty}^{\infty} |a_{k,j}(z)|^2 g(z)^{n-1} \sqrt{1+g'(z)^2}\, dz. \qquad (4.0.6)\end{aligned}$$

Therefore Theorem 2.3 will be a consequence of the following fact:

Lemma 4.1. *Given any sequence of positive indices $\{\nu_j\}$ with $\nu_j \geq \frac{n-2}{2}$ for all j and Schwartz functions a_j, the following inequality holds:*

$$\begin{aligned}\int_0^\infty \rho^{-q\frac{n-2}{2}+n-1} \left(\sum_j \int_{-\infty}^{\infty} |g(z)|^n \left| 1+(g'(z))^2 \right| |a_j(z)|^2 \left| J_{\nu_j}(\rho g(z)) \right|^2 dz \right)^{\frac{q}{2}} d\rho \\ \lesssim \left(\sum_j \int_{-\infty}^{\infty} |g(z)|^{n-1} \left(1+(g'(z))^2 \right)^{\frac{1}{2}} |a_j(z)|^2 dz \right)^{\frac{q}{2}}, \quad (4.0.7)\end{aligned}$$

for $q > \frac{2n}{n-1}$.

Remark 4.1. Taking into account the hypothesis about g we will look for estimates depending upon $A = \|g\|_\infty$ and $B = \|g'\|_\infty$. It is also easy to see that we can reduce ourselves to consider the sums over the family of indices $\{\nu_j\}_{j=1}^\infty$ such that $\nu_j \geq \frac{n-2}{2}$. Therefore it is enough to show

$$\begin{aligned}\int_0^\infty \rho^{-q\frac{n-2}{2}+n-1} \left(\sum_j \int_{-\infty}^{\infty} |b_j(z)|^2 \left| J_{\nu_j}(\rho g(z)) \right|^2 dz \right)^{\frac{q}{2}} d\rho \\ \lesssim \left(\sum_j \int_{-\infty}^{\infty} |b_j(z)|^2 dz \right)^{\frac{q}{2}} \quad (4.0.8)\end{aligned}$$

for a family of smooth functions $\{b_j\}_j$ and indexes $\nu_j \geq \frac{n-2}{2}$.

In order to show (4.0.8) we need a sharp control of the decay of Bessel functions. Once more, these estimates are detailed in Lemma A.1 of Appendix A.1.

Proof of Lemma 4.1 . To prove inequality (4.0.8) we shall first decompose the ρ-integration in dyadic parts: $[0, \infty) = [0, 1) \bigcup \cup_{n=0}^\infty [2^n, 2^{n+1})$.

$$\begin{aligned}\int_0^1 \rho^{-q\frac{n-2}{2}+n-1} \left(\sum_j \int_{-\infty}^{\infty} |b_j(z)|^2 \left| J_{\nu_j}(\rho g(z)) \right|^2 dz \right)^{\frac{q}{2}} d\rho \\ + \sum_M \int_M^{2M} \rho^{-q\frac{n-2}{2}+n-1} \left(\sum_j \int_{-\infty}^{\infty} |b_j(z)|^2 \left| J_{\nu_j}(\rho g(z)) \right|^2 dz \right)^{\frac{q}{2}} d\rho, \quad (4.0.9)\end{aligned}$$

where $M = 2^m$, $m = 0, 1, \ldots$

For the lower integrand, we have the following splitting:

$$\begin{aligned}\int_0^1 \rho^{-q\frac{n-2}{2}+n-1} [\ldots]^{\frac{q}{2}} \, d\rho &= \int_0^{\frac{1}{A}} \rho^{-q\frac{n-2}{2}+n-1} [\ldots]^{\frac{q}{2}} \, d\rho + \int_{\frac{1}{A}}^1 \rho^{-q\frac{n-2}{2}+n-1} [\ldots]^{\frac{q}{2}} \, d\rho \\ &= I + II.\end{aligned}$$

In order to bound I we invoke Minkowski's inequality and property 5. of Lemma A.1.

$$\begin{aligned} I &\lesssim \left[\int_{-\infty}^{\infty} \sum_j \left(\int_0^{\frac{1}{A}} \left\{ \rho^{-(n-2)+\frac{2}{q}(n-1)} \left|b_j(z)\right|^2 \left|J_{\nu_j}(\rho z)\right|^2 \right\}^{\frac{q}{2}} d\rho \right)^{\frac{2}{q}} dz \right]^{\frac{q}{2}} . \\ &\leq \left[\int_{-\infty}^{\infty} \sum_j \left|b_j(z)\right|^2 A^{2\nu_j} \left(\int_0^{\frac{1}{A}} \rho^{-q\frac{n-2}{2}+(n-1)+q\nu_j} d\rho \right)^{\frac{2}{q}} dz \right]^{\frac{q}{2}} , \end{aligned}$$

where $A = \|g\|_\infty$. Since the sum is taken over all $\nu_j \geq \frac{n-2}{2}$, the inner integrand is well defined and we can bound

$$I \lesssim A^{q\frac{n-1}{2}-n} \left[\sum_j \int_{-\infty}^{\infty} \left|b_j(z)\right|^2 dz \right]^{\frac{q}{2}} . \tag{4.0.10}$$

The second part is similarly bounded

$$II \lesssim \left(1 + A^{q\frac{n-1}{2}-n}\right) \left[\sum_j \int_{-\infty}^{\infty} \left|b_j(z)\right|^2 dz \right]^{\frac{q}{2}} . \tag{4.0.11}$$

Then Lemma 4.1 will be a consequence of the following claim:

Claim 4.1. For all $q > 4$, the following inequality holds true

$$\begin{aligned} &\int_M^{2M} \rho \left(\sum_j \int_{-\infty}^{\infty} \left|b_j(z)\right|^2 \left|J_{\nu_j}(\rho g(z))\right|^2 dz \right)^{\frac{q}{2}} d\rho \\ &\qquad \lesssim M^{\frac{4-q}{2}} \left(\int_{-\infty}^{\infty} \sum_j \left|b_j(z)\right|^2 dz \right)^{\frac{q}{2}} . \end{aligned} \tag{4.0.12}$$

Indeed, if $q > 4$ we need only to note that

$$\begin{aligned} &\int_M^{2M} \rho^{-q\frac{n-2}{2}+n-1} \left(\sum_j \int_{-\infty}^{\infty} \left|b_j(z)\right|^2 \left|J_{\nu_j}(\rho g(z))\right|^2 dz \right)^{\frac{q}{2}} d\rho \\ &\qquad \lesssim M^{(n-2)\left(-\frac{q}{2}+1\right)} \int_M^{2M} \rho \left(\sum_j \int_{-\infty}^{\infty} \left|b_j(z)\right|^2 \left|J_{\nu_j}(\rho g(z))\right|^2 dz \right)^{\frac{q}{2}} d\rho, \end{aligned}$$

invoke our claim and sum over all dyadic intervals in (4.0.9):

$$\sum_m \int_{2^m}^{2^{m+1}} \rho^{-q\frac{n-2}{2}+n-1} \left(\sum_j \int_{-\infty}^{\infty} |b_j(z)|^2 \left|J_{\nu_j}(\rho g(z))\right|^2 dz \right)^{\frac{q}{2}} d\rho$$
$$\lesssim \sum_m 2^{m(n-2)\left(-\frac{q}{2}+1\right)+m\frac{4-q}{2}} \left(\int_{-\infty}^{\infty} \sum_j |b_j(z)|^2 dz \right)^{\frac{q}{2}}. \quad (4.0.13)$$

It is then a simple matter to check that the exponent is negative for $q > \frac{2n}{n-1}$.

If the exponent q is however smaller, $\frac{2n}{n-1} < q \leq 4$, we need to use an extra trick. Note that equation (4.0.12) implies

$$\int_M^{2M} \left(\sum_j \int_{-\infty}^{\infty} |b_j(z)|^2 \left|J_{\nu_j}(\rho g(z))\right|^2 dz \right)^{\frac{q_1}{2}} d\rho \lesssim M^{1-\frac{q_1}{2}} \left(\int_{-\infty}^{\infty} \sum_j |b_j(z)|^2 dz \right)^{\frac{q_1}{2}},$$

for all $q_1 > 4$. Then using Hölder's inequality and the previous inequality,

$$\int_M^{2M} \left(\sum_j \int_{-\infty}^{\infty} |b_j(z)|^2 \left|J_{\nu_j}(\rho g(z))\right|^2 dz \right)^{\frac{q}{2}} d\rho$$
$$\lesssim M^{1-\frac{q}{q_1}} \left(\int_M^{2M} \left(\sum_j \int_{-\infty}^{\infty} |b_j(z)|^2 \left|J_{\nu_j}(\rho g(z))\right|^2 dz \right)^{\frac{q_1}{2}} d\rho \right)^{\frac{q}{q_1}}.$$

Therefore, summing over all intervals, we obtain

$$\sum_m \int_{2^m}^{2^{m+1}} \rho^{-q\frac{n-2}{2}+n-1} \left(\sum_j \int_{-\infty}^{\infty} |b_j(z)|^2 \left|J_{\nu_j}(\rho g(z))\right|^2 dz \right)^{\frac{q}{2}} d\rho$$
$$\lesssim \sum_m 2^{m\left\{-q\frac{n-2}{2}+n-1+1-\frac{q}{2}\right\}} \left(\int_{-\infty}^{\infty} \sum_j |b_j(z)|^2 dz \right)^{\frac{q}{2}},$$

where the exponent $-q\frac{n-1}{2}+n$ is negative for all $q > \frac{2n}{n-1}$.

We now turn our attention to prove Claim 4.1. Let us split each dyadic integrand in (4.0.9) in three parts corresponding to the different ranges of control of Bessel functions.

$$\begin{aligned}
&\int_M^{2M} \rho \left(\int_{-\infty}^{\infty} \sum_{\nu_j \in I^0} |b_j(z)|^2 \left| J_{\nu_j}(\rho g(z)) \right|^2 dz \right)^{\frac{q}{2}} d\rho \\
&+ \int_M^{2M} \rho \left(\int_{-\infty}^{\infty} \sum_{\nu_j \in I^c} |b_j(z)|^2 \left| J_{\nu_j}(\rho g(z)) \right|^2 dz \right)^{\frac{q}{2}} d\rho \\
&+ \int_M^{2M} \rho \left(\int_{-\infty}^{\infty} \sum_{\nu_j \in I^\infty} |b_j(z)|^2 \left| J_{\nu_j}(\rho g(z)) \right|^2 dz \right)^{\frac{q}{2}} d\rho \\
&= \sum_M \left(I_M^0 + I_M^c + I_M^\infty \right),
\end{aligned}$$

where $I_M^0 = [0, Mg(z)/2)$, $I_M^c = [Mg(z)/2, 4Mg(z))$, and $I_M^\infty = [4Mg(z), \infty)$.

Recall that if $2k < r$, $|J_k(r)| \le r^{-1/2}$; in I_M^0 we have $2\nu_j < Mg(z) < \rho g(z)$, hence

$$\begin{aligned}
I_M^0 &\le A^{-\frac{q}{2}} \int_M^{2M} \rho^{1-\frac{q}{2}} \left(\int_{-\infty}^{\infty} \sum_{\nu_j \in I^0} |b_j(z)|^2 dz \right)^{\frac{q}{2}} d\rho \\
&\le A^{-\frac{q}{2}} M^{\frac{4-q}{2}} \left(\int_{-\infty}^{\infty} \sum_{\nu_j} |b_j(z)|^2 dz \right)^{\frac{q}{2}}. \qquad (4.0.14)
\end{aligned}$$

Similarly, I_M^∞ is also easily bounded as if $k > 2r$, $|J_k(r)| \le k^{-1}$, and in I_M^∞, $k > 4Mg(z) > 2\rho g(z)$. Furthermore, since $\rho g(z) > 1$, $(\rho g(z))^{-2} < (\rho g(z))^{-1}$ and, in I_M^∞, we have $|J_k(\rho g(z))|^2 \le (\rho g(z))^{-1}$. This shows that again

$$I_M^\infty \le A^{-\frac{q}{2}} M^{\frac{4-q}{2}} \left(\int_{-\infty}^{\infty} \sum_{\nu_j} |b_j(z)|^2 dz \right)^{\frac{q}{2}}. \qquad (4.0.15)$$

Finally, we need to work a little bit harder than in the previous cases to obtain a suitable estimate for I_M^c. First of all note that Minkowski's inequality yields

$$I_M^c \le \left[\int_{-\infty}^{\infty} \left\{ \int_M^{2M} \rho \left(\sum_{\nu_j \in I^c} |b_j(z)|^2 \left| J_{\nu_j}(\rho g(z)) \right|^2 \right)^{\frac{q}{2}} d\rho \right\}^{\frac{2}{q}} dz \right]^{\frac{q}{2}}. \qquad (4.0.16)$$

In I_M^c we want to use estimate (3) of Lemma A.1, we thus need to split the inner integral so that $\rho g(z) \sim \nu_j + \alpha \nu_j$ in the according range of α. Consider the family of sets

$$G_\alpha = \left[\frac{M}{2} + \alpha M^{\frac{1}{3}} g(z)^{-\frac{2}{3}}, \frac{M}{2} + (\alpha + 1) M^{\frac{1}{3}} g(z)^{-\frac{2}{3}} \right],$$

for $\alpha = 0, 1, 2, \ldots, \left[(Mg(z))^{\frac{2}{3}} \right]$, so that $\bigcup G_\alpha \supseteq [M, 2M]$ and in each interval $\rho g(z) \sim \nu_j + \alpha \nu_j^{\frac{1}{3}}$, and split (4.0.16) in the following way

$$I_M^c \lesssim \left[\int_{-\infty}^{\infty} \left\{ \sum_\alpha \int_{G_\alpha} \rho \left(\sum_{\nu_j \in I^c} |b_j(z)|^2 \left| J_{\nu_j}(\rho g(z)) \right|^2 \right)^{\frac{q}{2}} d\rho \right\}^{\frac{2}{q}} dz \right]^{\frac{q}{2}},$$

Let us also define

$$A_\beta = \sum_{\nu_j \in G_\beta} |b_j(z)|^2 .$$

We can then invoke Lemma A.1 and rearrange the sums to bound I_M^c by

$$\left[\int_{-\infty}^{\infty} \left\{ \sum_\alpha \int_{G_\alpha} \left(\sum_{\beta \leq \alpha} A_\beta \frac{1}{(|\alpha - \beta| + 1)^{1/2} M^{\frac{2}{3}} g(z)^{-\frac{4}{3}}} \right)^{\frac{q}{2}} \rho d\rho \right\}^{\frac{2}{q}} dz \right]^{\frac{q}{2}}$$
$$+ \left[\int_{-\infty}^{\infty} \left\{ \sum_\alpha \int_{G_\alpha} \left(\sum_{\beta \geq \alpha} A_\beta \frac{1}{(|\alpha - \beta| + 1)^2 M^{\frac{2}{3}} g(z)^{-\frac{4}{3}}} \right)^{\frac{q}{2}} \rho d\rho \right\}^{\frac{2}{q}} dz \right]^{\frac{q}{2}} .$$

Note that the second sum is easier to control than the first. We shall, therefore, focus on the first term, $I_{M,}^{c,1}$. Since the intervals G_α have length $M^{\frac{1}{3}} g(z)^{-\frac{2}{3}}$,

$$I_M^{c,1} \lesssim M^{\frac{4-q}{3}} A^{2\frac{(q-1)}{3}} \left[\int_{-\infty}^{\infty} \left\{ \sum_\alpha \left(\sum_{\beta \geq \alpha} A_\beta \frac{1}{(|\alpha - \beta| + 1)^2} \right)^{\frac{q}{2}} \right\}^{\frac{2}{q}} dz \right]^{\frac{q}{2}} .$$

Furthermore, using Young's inequality, since $q > 4$, taking $2/q = 1/s - 1/2$ we obtain

$$\begin{aligned} \sum_\alpha \left(\sum_{\beta \geq \alpha} A_\beta \frac{1}{(|\alpha - \beta| + 1)^2} \right)^{\frac{q}{2}} &\lesssim \left(\sum_\gamma A_\gamma^s \right)^{\frac{q}{2s}} \\ &\lesssim \left(\sum_\gamma A_\gamma \right)^{\frac{q}{2}} . \end{aligned}$$

We have thus showed that the central integrand I_M^c can also be bounded in the desired way;

$$I_M^{c,1} \lesssim A^{2\left(\frac{q-1}{3}\right)} M^{\frac{4-q}{3}} \left[\int_{-\infty}^{\infty} \sum_{k \in I_M^c} |a_k|^2 \, dz \right]^{\frac{q}{2}} . \tag{4.0.17}$$

□

Chapter 5

The Hilbert transform with an oscillatory phase

5.1 Introduction

In the introduction we have already introduced the multi-parameter operators

$$\mathcal{H}^P f(x,y,z) := \int_{\mathbb{R}^2} f(x-s, y-t, z-P(s,t)) \frac{ds}{s}\frac{dt}{t}, \tag{5.1.1}$$

and

$$T_\lambda^P f(x,y) = \int_{\mathbb{R}^2} \frac{e^{2\pi i\lambda P(s,t)}}{st} f(x-s, y-t)\, dsdt.$$

These operators arose naturally in the early work of E. Fabes and N. Rivière on parabolic partial differential equations [21], and quickly attracted the attention of the harmonic analysts given their connection with the Calderón-Zygmund theory. The L^p behaviour of the 1-parameter analogue of $\mathcal{H}^P$ and T_λ^P, defined in (2.0.12) and (2.0.13), has long been well understood. In [35] F. Ricci and E. M. Stein showed that, if we formally define H_1^Q as

$$\mathcal{H}_1^Q f(x,y) := \int_{\mathbb{R}} f(x-s, y-Q(s)) \frac{ds}{s}$$

for any real polynomial Q of one variable, $\mathcal{H}_1^Q$ extends to a bounded operator of $L^p(\mathbb{R}^2)$ to itself for $1<p<\infty$. They also extended this result to higher dimensions for single parameter Calderón-Zygmund kernels. However, still many fundamental questions related to the endpoint mapping properties of this operator remain open; namely the boundedness of $\mathcal{H}_1^Q : H^1(\mathbb{R}^2) \to L^1(\mathbb{R}^2)$ for some Hardy-type space H^1. Such properties have only been understood for the operator associated to the parabola $Q_0(s) = s^2$,

$$\mathcal{H}_1^{Q_0} f(x,y) := \int_{-\infty}^{\infty} f\left(x-s, y-s^2\right) \frac{ds}{s}.$$

M. Christ **[11]** showed that in this special case $\mathcal{H}_1^{Q_0} : H^1(\mathbb{R}^2) \to L^{1,\infty}(\mathbb{R}^2)$, where $H^1(\mathbb{R}^2)$ denotes the usual parabolic real Hardy space associated to the family of dilations $\delta_r(x) = \left(rx_1, r^2x_2\right)$.

Let us take a step back from this open problem and try to understand the L^2 theory of a general operator $\mathcal{H}_1^Q$. Plancherel's identity applied only to the last variable yields

$$\left\|\mathcal{H}_1^Q f\right\|_{L^2(\mathbb{R}^2)}^2 \int_{\mathbb{R}} \int_{\mathbb{R}} \left|\mathcal{F}_2\left(\mathcal{H}_1^Q f\right)(x,\lambda)\right|^2 d\lambda dx,$$

where $\mathcal{F}_2$ denotes the Fourier transform in the second variable. Further,

$$\begin{aligned}
\mathcal{F}_2\left(\mathcal{H}_1^Q f\right)(x,\lambda) &= \int_{\mathbb{R}} \mathcal{H}_1^Q f(x,y)\, e^{-2\pi i y\lambda} dy \\
&= \int_{\mathbb{R}} K(s) \int_{\mathbb{R}} f(x-s, y-Q(s))\, e^{-2\pi i y\lambda} dy ds,
\end{aligned}$$

and a change of variables $t = y - Q(s)$ yields

$$\begin{aligned}
\mathcal{F}_2(Tf)(x,\lambda) &= \int_{\mathbb{R}} K(s) \int_{\mathbb{R}} f(x-s,t)\, e^{-2\pi i\lambda(t+Q(s))} dt ds \\
&= \int_{\mathbb{R}} e^{-2\pi i\lambda Q(s)} K(s)\, \mathcal{F}_2(f)(x-s,\lambda)\, ds \\
&= T_\lambda^{Q,1}(\mathcal{F}_2 f_\lambda)(x),
\end{aligned}$$

where

$$f_\lambda(x) = f(x,\lambda).$$

In the last equality we have encountered the lower dimensional operator

$$T_\lambda^{Q,1} g(x) = \int_{\mathbb{R}} e^{-2\pi i\lambda Q(s)} K(s)\, g(x-s)\, ds.$$

Plancherel's theorem allows to understand the L^2 behaviour of $\mathcal{H}_1^Q$ through $T_\lambda^{Q,1}$, as

$$\left\|\mathcal{H}_1^Q f\right\|_2^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \left|T_\lambda^{Q,1}(\mathcal{F}_2 f_\lambda)(x)\right|^2 dx d\lambda.$$

Clearly, if one could show that

$$\left\|T_\lambda^{Q,1} g\right\|_2 \lesssim \|g\|_2$$

uniformly in λ, then

$$\left\|\mathcal{H}_1^Q f\right\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R}^2)}.$$

However, for general p, the uniform boundedness of $T_\lambda^{Q,1}$ does not imply that of $\mathcal{H}_1^Q$. In fact it is the other way around: a vector valued application of a theorem of K. de Leeuw [18] yields, for every $1 < p < \infty$,

$$\sup_\lambda \left\|T_\lambda^{Q,1}\right\|_{L^p(\mathbb{R}^1)\to L^p(\mathbb{R}^1)} \leq \left\|\mathcal{H}_1^Q\right\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)}, \tag{5.1.2}$$

and

$$\sup_\lambda \left\|T_\lambda^{Q,1}\right\|_{L^2(\mathbb{R}^1)\to L^2(\mathbb{R}^1)} = \left\|\mathcal{H}_1^Q\right\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)}. \tag{5.1.3}$$

This study may be similarly carried in the multi-parameter case to obtain the analogous inequalities for $\mathcal{H}^P$ and T_λ^P (2.0.17) and (2.0.18).

For the rest of the chapter we focus our attention around the more accessible operators $T_\lambda^{Q,1}$ and T_λ^P. Notice that, in the one parameter case, Y. Pan [32] showed that

$$T_\lambda^{Q,1} : H^1(\mathbb{R}) \to L^1(\mathbb{R}),$$

again uniformly in the coefficients of P, where $H^1(\mathbb{R})$ denotes the usual Hardy space. In the bi-parametric case, however, no endpoint behaviour has yet been proved. The last result presented in this thesis gives a partial solution to the behaviour of T_λ^P in these critical spaces. As seen in Corollary 2.1 the L^p theory of T_λ^P is highly dependent on the geometrical properties of the Newton diagram of P. Therefore we cannot expect any better for the rectangular hardy space $H^1(\mathbb{R}^2)$ defined in Definition 2.1.

Lemma 5.1. *Let T be a convolution type operator such that $T : H^1_{rec}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$, then T also maps $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.*

This Lemma implies the necessity condition in Theorem 2.4.

Before proceeding to the full proof the Theorem, let's first explore some possible *a priori* reductions.

5.2 A priori reductions

As usual, in order to prove Theorem 2.4, it is enough to show that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^2} \left|T_\lambda^P a_{I\times J}(x,y)\right| dxdy \leq C,$$

uniformly in λ, I and J, for every rectangular atom $a_{I\times J}$. Moreover, after a simple change of variables we can assume without loss of generality that the atom is centered at the origin. Further, note that the change of variables

$$\begin{cases} |I|\,\tilde{s} = s, \\ |J|\,\tilde{t} = t, \end{cases}$$

yields

$$\begin{aligned} T_\lambda^P a_{I\times J}(x,y) &= \int_{\mathbb{R}^2} \frac{e^{i\lambda P\left(x-|I|\tilde{s},y-|J|\tilde{t}\right)}}{\left(x-|I|\,\tilde{s}\right)\left(y-|J|\,\tilde{t}\right)} a_{I\times J}\left(|I|\,\tilde{s},|J|\,\tilde{t}\right)|I|\,|J|\,d\tilde{s}d\tilde{t} \\ &= |I|^{-1}\,|J|^{-1}\int_{\mathbb{R}^2} \frac{e^{i\lambda\tilde{P}\left(|I|^{-1}x-\tilde{s},|J|^{-1}y-\tilde{t}\right)}}{\left(|I|^{-1}\,x-s\right)\left(|J|^{-1}\,y-\tilde{t}\right)} b\left(\tilde{s},\tilde{t}\right)d\tilde{s}d\tilde{t}, \end{aligned}$$

where

$$\tilde{P}(x,y) = \sum_{(i,j)\in\Delta} c_{i,j}\,|I|^i\,|J|^j\,x^iy^j,$$

and

$$b(x,y) = |I|\,|J|\,a_{I\times J}\left(|I|\,x,|J|\,y\right).$$

Hence

$$T_\lambda^P a_{I\times J}(x,y) = |I|^{-1}|J|^{-1}\tilde{T}_\lambda^P b\left(|I|^{-1}x, |J|^{-1}y\right), \tag{5.2.1}$$

where

$$\tilde{T}_\lambda^P g(x,y) := p.v.\int_{\mathbb{R}^2}\frac{e^{i\lambda\tilde{P}(x-s,y-t)}}{(x-s)(y-t)}g(s,t)\,dsdt.$$

The L^1 norm of our operator evaluated at $a_{I\times J}$ now becomes

$$\begin{aligned}\int_{\mathbb{R}^2}\left|T_\lambda^P a_{I\times J}(x,y)\right|dxdy &= |I|^{-1}|J|^{-1}\int_{\mathbb{R}^2}\left|\tilde{T}_\lambda^P b\left(|I|^{-1}x,|J|^{-1}y\right)\right|dxdy\\ &= \int_{\mathbb{R}^2}\left|\tilde{T}_\lambda^P b(x,y)\right|dxdy,\end{aligned}$$

and thus in order to prove Theorem 2.4, it is enough to show that

$$\int_{\mathbb{R}^2}\left|\tilde{T}_\lambda^P b(x,y)\right|dxdy \le C,$$

uniformly in λ, I and J, where b is a rectangular atom centered at the origin and supported in the unit square. Indeed, it is a simple task to check that

1. $\mathrm{supp}(b)\subset[-1/2,1/2]\times[-1/2,1/2]$
2. $\int_{[-1/2,1/2]} b(x,y)\,dx = 0$, for all $y\in[-1/2,1/2]$
3. $\int_{[-1/2,1/2]} b(x,y)\,dy = 0$, for all $x\in[-1/2,1/2]$
4. $\|b\|_{L^2}\le 1$.

Further we may introduce a partition of unity inside the operator $\tilde{T}_\lambda^P$ as follows

$$\begin{aligned}\left\|\tilde{T}_\lambda^P b\right\|_{L^1} &= \int_{\mathbb{R}^2}\left|\sum_{(p,q)\in\mathbb{Z}^2}\int_{\mathbb{R}^2}e^{i\lambda\tilde{P}(x-s,y-t)}\frac{\psi\left(2^{-p}(x-s)\right)}{(x-s)}\frac{\psi\left(2^{-q}(y-t)\right)}{(y-t)}b(s,t)\,dsdt\right|dxdy\\ &= \int_{\mathbb{R}^2}\left|\sum_{(p,q)\in\mathbb{Z}^2}S_\lambda^{p,q}b(x,y)\right|dxdy, \end{aligned}\tag{5.2.2}$$

where $\psi\in C_0^\infty(\mathbb{R})$ is non-negative and such that $\psi(x)=1$ for all $|x|\in[1,2]$ and $\psi(x)=0$ for $|x|\in[\frac{1}{2},4]^c$ and $\sum_p\psi(2^{-p}x)=1$ for all $x\in\mathbb{R}\backslash\{0\}$.

Notation. Throughout this chapter any constant C will not depend on λ, I, or J, but might depend on the degree of P or its coefficients. We might specify the dependence of any constant C on additional parameters e.g. α by writing C_α. Notice that the value of C might differ from line to line.

5.3 The bounded region

First we take care of the bounded region near the origin

$$B = \left\{(p,q)\in\mathbb{Z}^2, 2^p < C_0, 2^q < C_0\right\},$$

for some large constant $C_0 > 1$ not depending on λ, $|I|$ and $|J|$. Then note that for $|x| \geq 5C_0$ and $|s| \leq 1/2$, it follows that $|x-s| > 4C_0$ and thus $\psi\left(2^{-p}(x-s)\right) = 0$. Therefore we have

$$\int_{\mathbb{R}^2} \left| \sum_{(p,q)\in B} S_\lambda^{p,q} b\left(x,y\right) \right| dxdy \lesssim \int_{\substack{|x|\leq 5C_0 \\ |y|\leq 5C_0}} \left| \sum_{(p,q)\in B} S_\lambda^{p,q} b\left(x,y\right) \right| dxdy,$$

and

$$\sum_{(p,q)\in B} S_\lambda^{p,q} b\left(x,y\right) = \int_{\mathbb{R}^2} e^{i\lambda\tilde{P}(x-s,y-t)} \frac{\chi\left(x-s\right)}{\left(x-s\right)} \frac{\chi\left(y-t\right)}{\left(y-t\right)} b\left(s,t\right) dsdt,$$

where $\chi \in C^\infty\left(\mathbb{R}\right)$ is a smooth cutoff centered at 0 of radius C_0. This term can be easily bounded by means of the Cauchy-Schwartz inequality and the well-understood L^2-theory. Note that the L^2 boundedness properties of the smoothened operator $\sum_{\substack{2^p<C_0 \\ 2^q<C_0}} S_\lambda^{p,q} b\left(x,y\right)$ are inherited from those of $\tilde{T}_\lambda$,

$$\left\| \sum_{(p,q)\in B} S_\lambda^{p,q} b \right\|_2 \leq \left\| \tilde{T}_\lambda \right\|_2.$$

Therefore, using the fact that P belongs to the class of admissible polynomials and (5.2.1),

$$\begin{aligned} \int_{\mathbb{R}^2} \left| \sum_{(p,q)\in B} S_\lambda^{p,q} b\left(x,y\right) \right| dxdy &\lesssim |I|\,|J|\,\left\|T_\lambda a_{I\times J}\right\|_2^2 \\ &\lesssim 1. \end{aligned} \tag{5.3.1}$$

It is important to note that in order to obtain bounds uniform in $|I|$ and $|J|$ we cannot rely on the boundedness properties of the localized Hilbert transform. Therefore it is crucial to impose that P belongs to the class of admissible polynomials.

5.4 The unbounded region: the single monomial case

In order to obtain a more accessible approach to the proof we first proceed to study the single monomial case $P\left(x,y\right) = c_{i,j}x^i y^j$ with $i \times j$ even. Therefore the operator $S_\lambda^{p,q}$ becomes

$$S_\lambda^{p,q} g\left(x,y\right) = \int_{\mathbb{R}^2} e^{i\tilde{\lambda}(x-s)^i(y-t)^j} \frac{\psi\left(2^{-p}\left(x-s\right)\right)}{\left(y-t\right)} \frac{\psi\left(2^{-q}\left(y-t\right)\right)}{\left(y-t\right)} g\left(s,t\right) dsdt, \tag{5.4.1}$$

with $\tilde{\lambda} = c_{i,j}\left|I\right|^i\left|J\right|^j$. Also, before proceeding to study the behaviour of the operator in the unbounded region $U = \left\{(p,q)\in\mathbb{R}^2, 2^p > C_0 \text{ or } 2^q > C_0\right\}$, note that it is enough to show that

$$\int_{\mathbb{R}^2} \left| \sum_{\substack{p\leq q \\ 2^q>C_0}} S_\lambda^{p,q} b\left(x,y\right) \right| dxdy \leq C, \tag{5.4.2}$$

as the complementary set of dyadic regions may be treated symmetrically.

We simplify the notation by defining the set $\tilde{U} \subset \mathbb{Z}^2$ as

$$\tilde{U} := \left\{(p,q) \in \mathbb{Z}^2, p \leq q, 2^q > C_0\right\}.$$

By using the fact that $\int_{\mathbb{R}} b\,(s,t)\,dst = 0$ for all $t \in \mathbb{R}$, we produce the following splitting

$$\int_{\mathbb{R}^2} \left| \sum_{(p,q)\in\tilde{U}} S_\lambda^{p,q} b\,(x,y) \right| dxdy \leq I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^2} \left| \sum_{(p,q)\in\tilde{U}} \int_{\mathbb{R}^2} \left\{ e^{i\tilde{\lambda}(x-s)^i (y-t)^j} - e^{i\tilde{\lambda}(x-s)^i y^j} \right\} \frac{\psi\left(2^{-p}\,(x-s)\right)}{(x-s)} \frac{\psi\left(2^{-q}\,(y-t)\right)}{(y-t)} b\,(s,t)\,dsdt \right| dxdy, \\ I_2 &= \int_{\mathbb{R}^2} \left| \sum_{(p,q)\in\tilde{U}} \int_{\mathbb{R}^2} e^{i\tilde{\lambda}(x-s)^i y^j} \left\{ \frac{\psi\left(2^{-q}\,(y-t)\right)}{y-t} - \frac{\psi\left(2^{-q} y\right)}{y} \right\} \frac{\psi\left(2^{-p}\,(x-s)\right)}{(x-s)} b\,(s,t)\,dsdt \right| dxdy. \quad (5.4.3) \end{aligned}$$

Let us first proceed to bound I_2 in a way that might not be the easiest but is valid for any polynomial P. We split the sum $\sum_{p\leq q} = \sum_{p\leq 0} + \sum_{0\leq p\leq q}$ so that $I_2 \leq I_2^1 + I_2^2$. Then, using the mean value theorem

$$\left| \frac{\psi\left(2^{-q}\,(y-t)\right)}{y-t} - \frac{\psi\left(2^{-q} y\right)}{y} \right| \lesssim 2^{-2q} \quad (5.4.4)$$

and taking into account the support of b,

$$\begin{aligned} I_2^2 &\leq \sum_{2^q > C_0} \sum_{0\leq p\leq q} 2^{-2q} 2^{-p} \\ &< C. \end{aligned}$$

We need to work a bit harder to bound the term I_2^1. Notice that after a few manipulations the Cauchy Schwartz inequality in the x variable yields

$$I_2^1 \leq \sum_{2^q > C_0} \int_{|y|\sim 2^q} \int_{\mathbb{R}} \left| \frac{\psi\left(2^{-q}\,(y-t)\right)}{y-t} - \frac{\psi\left(2^{-q} y\right)}{y} \right| \left\| T_\lambda^{y,t} b \right\|_{L^2(\mathbb{R})} \left(\sum_{p\leq 0} \int_{|x|\sim 2^p} dx \right)^{\frac{1}{2}} dtdy,$$

where

$$T_\lambda^{y,t} b\,(x) := \int_{\mathbb{R}} e^{i\tilde{\lambda}(x-s)^i y^j} \sum_{p\leq 0} \frac{\psi\left(2^{-p}\,(x-s)\right)}{x-s} b\,(s,t)\,ds. \quad (5.4.5)$$

Notice that given the properties of the rectangular atom b, $b\,(s,t) \in L_s^2\,(\mathbb{R})$ for every t. Further, $T_\lambda^{y,t}$ is corresponds to a smoothened one-dimensional Hilbert transform with an oscillatory phase, and thus it is bounded in L^2 uniformly in the coefficient $c_i = \tilde{\lambda} y^j$ of the polynomial $Q\,(x) = c_i x^i$ as long as i is even, cf. [36]. Hence,

$$I_2 \leq \sum_{2^q > C_0} \int_{|y|\sim 2^q} \int_{\mathbb{R}} \left| \frac{\psi\left(2^{-q}\,(y-t)\right)}{y-t} - \frac{\psi\left(2^{-q} y\right)}{y} \right| \left(\int_{\mathbb{R}} |b\,(s,t)|^2\,ds \right)^{\frac{1}{2}} dtdy.$$

Finally, we appeal once more the mean value theorem (5.4.4) and Cauchy Schwartz inequality in t to bound,

$$\begin{aligned} I_2 &\leq \sum_{2^q > C_0} 2^{-q} \|b\|_{L^2(\mathbb{R}^2)} \\ &\lesssim C. \end{aligned}$$

Next, in order to produce appropriate bounds for the integral I_1, note that

$$I_1 \leq \sum_{(p,q)\in \tilde{U}} I_{p,q},$$

where $I_{p,q}$ is of the form

$$\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} \left\{ e^{i\tilde{\lambda}(x-s)^i(y-t)^j} - e^{i\tilde{\lambda}(x-s)^i y^j} \right\} \frac{\psi\left(2^{-p}(x-s)\right)}{x-s} \frac{\psi\left(2^{-q}(y-t)\right)}{y-t} b(s,t)\, dsdt \right| dxdy.$$

We will provide several bounds for $I_{p,q}$ and then interpolate between them. First, we note that the mean value theorem in the second variable of the polynomial yields

$$\left| e^{i\tilde{\lambda}(x-s)^i(y-t)^j} - e^{i\tilde{\lambda}(x-s)^i y^j} \right| \lesssim \tilde{\lambda} \frac{1}{|c|} |x-s|^i |c|^j |t|,$$

with $|c| \sim |y-t|$. Therefore

$$\begin{aligned} I_{p,q} &\lesssim \tilde{\lambda} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |x-s|^i |y-t|^j \frac{\psi\left(2^{-p}(x-s)\right)}{|x-s|} \frac{\psi\left(2^{-q}(y-t)\right)}{|y-t|^2} |b(s,t)|\, |t|\, dsdtdxdy \\ &\lesssim \tilde{\lambda} 2^{pi+q(j-1)}. \end{aligned} \tag{5.4.6}$$

This estimate is however not completely satisfactory and we need to produce some alternative bound.

Let us go back to the definition of $I_{p,q}$:

$$\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} \left\{ e^{i\tilde{\lambda}(x-s)^i(y-t)^j} - e^{i\tilde{\lambda}(x-s)^i y^j} \right\} \frac{\psi\left(2^{-p}(x-s)\right)}{x-s} \frac{\psi\left(2^{-q}(y-t)\right)}{y-t} b(s,t)\, dsdt \right| dxdy.$$

and note that we may split $I_{p,q}$ into

$$\begin{aligned} I_{p,q}^{(1)} + I_{p,q}^{(2)} = &\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} e^{i\tilde{\lambda}(x-s)^i(y-t)^j} \frac{\psi\left(2^{-p}(x-s)\right)}{(x-s)} \frac{\psi\left(2^{-q}(y-t)\right)}{(y-t)} b(s,t)\, dsdt \right| dxdy \\ &+ \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} e^{i\tilde{\lambda}(x-s)^i y^j} \frac{\psi\left(2^{-p}(x-s)\right)}{(x-s)} \frac{\psi\left(2^{-q}(y-t)\right)}{(y-t)} b(s,t)\, dsdt \right| dxdy \end{aligned}$$

We will focus on studying $I_{p,q}^{(1)}$ and the $I_{p,q}^{(2)}$ will follow similarly. First note that

$$I_{p,q}^{(1)} = \int_{\mathbb{R}^2} \left| T_{\tilde{\lambda},p,q} b(x,y) \right| dxdy,$$

with

$$T_{\tilde{\lambda},p,q} g(x,y) = \int_{\mathbb{R}^2} \varphi(s,t)\, e^{i\tilde{\lambda}(x-s)^i(y-t)^j} \frac{\psi\left(2^{-p}(x-s)\right)}{(x-s)} \frac{\psi\left(2^{-q}(y-t)\right)}{(y-t)} g(s,t)\, dsdt, \tag{5.4.7}$$

where $\varphi \in C_c^\infty(\mathbb{R}^2)$ with $\varphi \equiv 1$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]^2$ and $\varphi \equiv 0$ outside $[-1,1]^2$. A simple use of the Cauchy-Schwarz inequality makes it clear that we are interested in bounds of $\left\|T_{\tilde\lambda,p,q}b\right\|_{L^2}$,

$$\sum_{(p,q)\in\tilde U} I_{p,q}^{(1)} \le \sum_{(p,q)\in\tilde U} 2^{\frac{p+q}{2}} \left\|T_{\tilde\lambda,p,q}b\right\|_{L^2}.$$

Let us take a leap forward and assume that we are able to obtain the following:

Lemma 5.2. *Let $T_{\tilde\lambda,p,q}$ be the operator defined in* (5.4.7) *with $(p,q)\in\tilde U$, then $T_{\tilde\lambda,p,q}$ is a bounded operator from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ and satisfies the following bounds:*

$$\left\|T_{\tilde\lambda,p,q}b\right\|_{L^2} \lesssim \begin{cases} \frac{1}{2^{\frac{q}{2}}}\left(\frac{1}{\tilde\lambda 2^{pi+qj}}\right)^\delta \|b\|_{L^2}, & p\le 0,\\ \frac{1}{2^{\frac{p+q}{2}}}\left(\frac{1}{\tilde\lambda 2^{p(i-1)+qj}}\right)^\delta \|b\|_{L^2}, & p>0,\end{cases} \tag{5.4.8}$$

for some $0<\delta<1$.

Then, following the definition of $I_{p,q}^{(k)}$, we have

$$I_{p,q} \le \begin{cases} \left(\frac{1}{\tilde\lambda 2^{pi+qj}}\right)^\delta, & p\le 0,\\ \left(\frac{1}{\tilde\lambda 2^{p(i-1)+qj}}\right)^\delta, & p>0.\end{cases} \tag{5.4.9}$$

We will show the validity of this lemma in Section 5.4.3. We now show that these estimates are enough to finish the proof of the main Theorem.

5.4.1 When $p>0$

First we take care of the sum over $p>0$, that is

$$\sum_{\substack{p>0\\(p,q)\in\tilde U}} I_{p,q} = \sum_{\substack{p>0\\(p,q)\in\tilde U\\ \tilde\lambda 2^{qL}\le 1}} I_{p,q} + \sum_{\substack{p>0\\(p,q)\in\tilde U\\ \tilde\lambda 2^{qL}>1}} I_{p,q}, \tag{5.4.10}$$

where $L=i+j-1$. Then the first summand is easily bounded by using (5.4.6).

We need to work a bit harder to show that the second summand is also bounded. In fact we need to interpolate both bounds obtained in (5.4.6) and (5.4.9) by choosing a constant $0<\theta<1$ satisfying

$$\frac{i-1}{i}\delta < \frac{\theta}{1-\theta} < \delta. \tag{5.4.11}$$

Then we have

$$\begin{aligned} I_{p,q} &\lesssim \frac{\left(\tilde\lambda 2^{pi+q(j-1)}\right)^\theta}{\left(\tilde\lambda 2^{p(i-1)+qj}\right)^{\delta(1-\theta)}}\\ &= \frac{2^{p\{i\theta-(i-1)\delta(1-\theta)\}}}{\tilde\lambda^{\delta(1-\theta)-\theta}2^{q\{j\delta(1-\theta)-(j-1)\theta\}}}. \end{aligned}$$

Furthermore, since θ has been chosen to satisfy (5.4.11), it follows that

$$i\theta - (i-1)\,\delta\,(1-\theta) > 0.$$

Thus since $p \leq q$ if $(p,q) \in \tilde{U}$,

$$\begin{aligned}\sum_{p\leq q} I_{p,q} &\lesssim \frac{2^{q\{i\theta-(i-1)\delta(1-\theta)\}}}{\tilde{\lambda}^{\delta(1-\theta)-\theta}2^{q\{j\delta(1-\theta)-(j-1)\theta\}}} \\ &= \left(\frac{1}{\tilde{\lambda}2^{qL}}\right)^{\delta(1-\theta)-\theta}.\end{aligned}$$

Therefore, given our choice of θ, $\delta\,(1-\theta)-\theta > 0$ and it follows that

$$\sum_{\tilde{\lambda}2^{qL}>1}\left(\frac{1}{\tilde{\lambda}2^{qL}}\right)^{\delta(1-\theta)-\theta} \lesssim 1.$$

We have finally obtained

$$\sum_{\substack{p>0\\(p,q)\in\tilde{U}}} I_{p,q} \lesssim 1.$$

5.4.2 When $p \leq 0$

We produce the following splitting

$$\sum_{\substack{p\leq 0\\(p,q)\in\tilde{U}}} I_{p,q} = \sum_{\substack{p\leq 0\\(p,q)\in\tilde{U}\\ \tilde{\lambda}2^{pi}\leq 1}} I_{p,q} + \sum_{\substack{p\leq 0\\(p,q)\in\tilde{U}\\ \tilde{\lambda}2^{pi}>1}} I_{p,q}. \tag{5.4.12}$$

Then the second summand is easily dealt with by means of the bound (5.4.9),

$$\begin{aligned}\sum_{\substack{p\leq 0\\(p,q)\in\tilde{U}\\ \tilde{\lambda}2^{pi}>1}} I_{p,q} &\lesssim \sum_{2^q>C_0}\frac{1}{2^{\delta qj}}\sum_{\tilde{\lambda}2^{pi}>1}\left(\frac{1}{\tilde{\lambda}2^{pi}}\right)^{\delta} \\ &\lesssim 1,\end{aligned}$$

since $\delta > 0$.

Finally, to bound the first summand of (5.4.12), again we need to interpolate between both estimates (5.4.6) and (5.4.9). First, choose $0 < \omega_1 < 1$ satisfying

$$(j-1)\,\omega_1 = j\delta\,(1-\omega_1)\,,$$

say $\omega_1 = \frac{j\delta}{j\delta+(j-1)}$, and note that

$$\left|e^{i\lambda\tilde{P}(x-s,y-t)} - e^{i\lambda\tilde{P}(x-s,y)}\right| \lesssim \lambda\frac{1}{|c|}\left|\tilde{P}\,(x-s,c)\right||t|\,,$$

with $|c| \sim |y-t|$, and thus

$$I_{p,q} \lesssim \lambda\sum_{(m,n)\in\Delta}|c_{m,n}|\,|I|^m\,|J|^n\,2^{pm+q(n-1)}.$$

$$\begin{aligned}
I_{p,q} &\leq \frac{\left(\tilde{\lambda}2^{pi+q(j-1)}\right)^{\omega_1}}{\left(\tilde{\lambda}2^{pi+qj}\right)^{\delta(1-\omega_1)}} \\
&= \frac{\left(\tilde{\lambda}2^{pi}\right)^{\omega_1-\delta(1-\omega_1)}}{2^{q\{j\delta(1-\omega_1)-(j-1)\omega_1\}}} \\
&= \left(\tilde{\lambda}2^{pi}\right)^{\omega_1-\delta(1-\omega_1)}. \qquad (5.4.13)
\end{aligned}$$

On the other hand choose $0 < \omega_2 < 1$ satisfying

$$\omega_2 = \delta\left(1-\omega_2\right),$$

say $\omega_2 = \frac{\delta}{1+\delta}$. Then interpolating the same estimates we obtain

$$\begin{aligned}
I_{p,q} &\leq \frac{\left(\tilde{\lambda}2^{pi}\right)^{\omega_2-\delta(1-\omega_2)}}{2^{q\{j\delta(1-\omega_2)-(j-1)\omega_2\}}} \\
&= \left(\frac{1}{2^{qj}}\right)^{\omega_2}. \qquad (5.4.14)
\end{aligned}$$

Finally, interpolating both estimates (5.4.13) and (5.4.14), we obtain

$$\sum_{\tilde{\lambda}2^{pj}<1}\sum_{2^q>C_0}\frac{\left(\tilde{\lambda}2^{pi}\right)^{\frac{\omega_1-\delta(1-\omega_1)}{2}}}{2^{\frac{q\omega_2}{2}}} < C,$$

since both $\omega_1-\delta\left(1-\omega_1\right) > 0$ and $\omega_2 > 0$.

5.4.3 $T^*_{\lambda,p,q}T_{\lambda,p,q}$ estimates

In this section we will be devoted prove Lemma 5.2. We need to produce estimates for the operator

$$T_{\lambda,p,q}g\left(x,y\right) := \int_{\mathbb{R}^2}\varphi\left(s,t\right)e^{i\tilde{\lambda}(x-s)^i(y-t)^j}\frac{\psi\left(2^{-p}\left(x-s\right)\right)}{\left(x-s\right)}\frac{\psi\left(2^{-q}\left(y-t\right)\right)}{\left(y-t\right)}g\left(s,t\right)dsdt,$$

where $\varphi \in C_c^\infty\left(\mathbb{R}^2\right)$ with $\varphi \equiv 1$ on $\left[-\frac{1}{2},\frac{1}{2}\right]^2$ and $\varphi \equiv 0$ outside $\left[-1,1\right]^2$. It is straightforward to check that the kernel of $T^*_{\lambda,p,q}T_{\lambda,p,q}$ is given by

$$L\left(s,t,u,v\right) = \varphi\left(s,t\right)\varphi\left(u,v\right)\int_{\mathbb{R}^2}e^{i\tilde{\lambda}\tilde{\phi}(x,y,s,t,u,v)}G\left(x,y,s,t,u,v\right)dxdy,$$

where

$$G\left(x,y,s,t,u,v\right) = \frac{\psi\left(2^{-p}\left(x-s\right)\right)\psi\left(2^{-p}\left(x-u\right)\right)\psi\left(2^{-q}\left(y-t\right)\right)\psi\left(2^{-q}\left(y-v\right)\right)}{\left(x-s\right)\left(x-u\right)\left(y-t\right)\left(y-v\right)}$$

and

$$\tilde{\phi}\left(x,y,s,t,u,v\right) = \left(x-s\right)^i\left(y-t\right)^j - \left(x-u\right)^i\left(y-v\right)^j.$$

After a simple change of variables we have

$$L(s,t,u,v) = \varphi(s,t)\,\varphi(u,v)\,\frac{1}{2^{p+q}} \int_{(x,y)\in B} e^{i\tilde{\lambda}2^{pi+qj}\tilde{\phi}\left(x,y,2^{-p}s,2^{-q}t,2^{-p}u,2^{-q}v\right)}\tilde{G}(x,y,s,t,u,v)\,dxdy,$$

where

$$\tilde{G}(x,y,s,t,u,v) = \frac{\psi(x-2^{-p}s)\,\psi(x-2^{-p}u)\,\psi(y-2^{-q}t)\,\psi(y-2^{-q}v)}{(x-2^{-p}s)(x-2^{-p}u)(y-2^{-q}t)(y-2^{-q}v)},$$

and $B = \left\{(x,y)\in\mathbb{R}^2, |x-2^{-p}s| \sim |x-2^{-p}u| \sim |y-2^{-p}t| \sim |y-2^{-p}v| \sim 1\right\}$ and

Note that a simple interpolation between

$$\sup_{u,v}\int |L(s,t,u,v)|\,dsdt \lesssim \begin{cases} \frac{1}{2^q}\left(\frac{1}{\tilde{\lambda}2^{pi+qj}}\right)^{\delta}, & p\le 0,\\ \frac{1}{2^q}\left(\frac{1}{\tilde{\lambda}2^{p(i-1)+qj}}\right)^{\delta}, & p>0,\end{cases} \tag{5.4.15}$$

and

$$\sup_{s,t}\int |L(s,t,u,v)|\,dudv \lesssim \begin{cases} \frac{1}{2^q}\left(\frac{1}{\tilde{\lambda}2^{pi+qj}}\right)^{\delta}, & p\le 0,\\ \frac{1}{2^q}\left(\frac{1}{\tilde{\lambda}2^{p(i-1)+qj}}\right)^{\delta}, & p>0,\end{cases}$$

produces the L^2 estimates of (5.4.8), and thus the proof is finished.

We denote by ϕ the oscillatory phase

$$\phi(x,y) = \tilde{\lambda}2^{pi+qj}\left\{\left(x-2^{-p}s\right)^i\left(y-2^{-q}t\right)^j - \left(x-2^{-p}u\right)^i\left(y-2^{-q}v\right)^j\right\},$$

and look for upper bounds of derivatives of the oscillatory phase in order to apply the n-dimensional *van der Corput lemma*, Proposition A.5. In fact we are interested in the $i+j-1$ derivative

$$\frac{\partial^{i+j-1}\phi}{\partial x^{i-1}\partial y^j}(x,y) = \tilde{\lambda}2^{pi+qj}\,(i-1)!j!2^{-p}\,(u-s)\,.$$

Let us take care only of the case $p\le 0$ as the other one will be similarly obtained. Since $p\le 0$ note that $L=0$ unless $|u-s|<2^p$. Then, note that the the estimate on $\frac{\partial^{i+j-1}\phi}{\partial x^{i-1}\partial y^j}$ together with the van der Corput estimate yields

$$\begin{aligned} \sup_{u,v}\int_B |L(s,t,u,v)|\,dsdt &\lesssim \sup_{u,v}\frac{1}{2^{p+q}}\left(\frac{1}{\left|\tilde{\lambda}\right|2^{p(i-1)+qj}}\right)^{\frac{1}{i+j-1}}\int_{|u-s|<2^p}|u-s|^{-\frac{1}{i+j-1}}\,ds \\ &\le \frac{2^{p\left(1-\frac{1}{i+j-1}\right)}}{2^{p+q}}\left(\frac{1}{\left|\tilde{\lambda}\right|2^{p(i-1)+qj}}\right)^{\frac{1}{i+j-1}} \\ &\le \frac{1}{2^q}\left(\frac{1}{\tilde{\lambda}2^{pi+qj}}\right)^{\frac{1}{i+j-1}}. \end{aligned} \tag{5.4.16}$$

An analogous procedure can be performed when the supremum is taken over s and t. In the case $p > 0$, we need to notice that now $L = 0$ unless $|u - s| < 1$ and make use of the fact that $2^{-p} < 1$. This concludes the proof of the single monomial case.

5.5 The unbounded region: the general polynomial case

We now proceed to study the boundedness of the operator $\sum_{(p,q)\in U} S_\lambda^{p,q}$ defined for a general polynomial as in (5.2.2)

$$S_\lambda^{p,q} g(x,y) = \int_{\mathbb{R}^2} e^{i\lambda \tilde{P}(x-s,y-t)} \frac{\psi\left(2^{-p}(x-s)\right)}{(y-t)} \frac{\psi\left(2^{-q}(y-t)\right)}{(y-t)} g(s,t)\, dsdt,$$

where as U corresponds to the unbounded region $\left\{(p,q) \in \mathbb{Z}^2, 2^p > C_0 \text{ or } 2^q > C_0\right\}$. As in the case of a single monomial, by symmetry it is enough to consider the region $\tilde{U} = \left\{(p,q) \in \mathbb{Z}^2, p \le q, 2^q > C_0\right\}$. Motivated by the approach of [5] we will split $U = \cup_{(i,j)\in\Delta} A_{i,j}$, where the in each region $A_{i,j}$ the monomial $x^i y^j$ dominates the rest, that is

$$A_{i,j} = \left\{(p,q) \in \tilde{U}, |c_{i,j}| \left(|I|\, 2^p\right)^i \left(|J|\, 2^q\right)^j \ge |c_{m,n}| \left(|I|\, 2^p\right)^m \left(|J|\, 2^q\right)^n, \forall (m,n) \in \Delta \backslash \{(i,j)\}\right\}.$$

Therefore

$$\left\| \sum_{(p,q)\in\tilde{U}} S_\lambda^{p,q} b \right\|_1 \le \sum_{(i,j)\in\Delta} \left\| \sum_{(p,q)\in A_{i,j}} S_\lambda^{p,q} b \right\|_1,$$

Since the constant in the desired bound may depend on the degree of the polynomial, it is enough to show that

$$\int_{\mathbb{R}^2} \left| \sum_{(p,q)\in A_{i,j}} S_\lambda^{p,q} b(x,y) \right| dxdy \le C, \tag{5.5.1}$$

for all $(i,j) \in \Delta$. Our hope is that, in each region $A_{i,j}$, $S_\lambda^{p,q}$ will behave like its analogous single monomial operator (5.4.1), which we know is bounded.

5.5.1 Region where a monomial dominates the rest tightly

Let us to further tweak the definition of the sets $A_{i,j}$. The region $(p,q) \in A_{i,j}$ corresponds to that where the monomial corresponding to (i,j) is dominating all of the others. However we want to further impose that this monomial is *tightly dominating* the others, that is

$$A_{i,j}^{(1)} = \left\{(p,q) \in A_{i,j}, \sup_{(m,n)\in\Delta\backslash\{(i,j)\}} |c_{m,n}| \left(|I|\, 2^p\right)^m \left(|J|\, 2^q\right)^n \ge \frac{1}{M} |c_{i,j}| \left(|I|\, 2^p\right)^i \left(|J|\, 2^q\right)^j \right\},$$

where M is a large constant depending only on $|\Delta|$. Suppose that the previous supremum is attained at $(i_0, j_0) \in \Delta\backslash\{(i,j)\}$. It then follows that

$$\log\left(\frac{1}{M}\left(2^q\, |J|\right)^{j-j_0} \left|\frac{c_{i,j}}{c_{i_0,j_0}}\right| |I|^{i-i_0}\right)^{\frac{1}{i-i_0}} \le p \le \log\left(\left(|J|\, 2^q\right)^{j-j_0} \left|\frac{c_{i,j}}{c_{i_0,j_0}}\right| |I|^{i-i_0}\right)^{\frac{1}{i-i_0}},$$

and thus we only have a few admissible p, for which

$$p = f(q) - L$$

$L = 0, \ldots, \frac{1}{i-i_0}\log M$. Notice that we may assume by symmetry that $i - i_0 > 0$.

Let us first consider the case $j = j_0$, for which

$$\log\left(\frac{1}{M}\left|\frac{c_{i,j}}{c_{i_0,j_0}}\right||I|^{i-i_0}\right)^{\frac{1}{i-i_0}} \leq p \leq \log\left(\left|\frac{c_{i,j}}{c_{i_0,j_0}}\right||I|^{i-i_0}\right)^{\frac{1}{i-i_0}}.$$

Therefore, since $p \sim \log c$

$$\int_{\mathbb{R}^2}\left|\sum_{\substack{p,q\in A_{i,j}^{(1)}\\ p\leq q\\ 2^q>C_0}} S_\lambda^{p,q} b(x,y)\right| dxdy$$
$$\leq M\int\int\left|\sum_{2^q>C_0}\int e^{iP(c,y-t)}\psi\left(2^{-q}(x-s)\right)b(s,t)\,dt\right| dxds$$

We have thus reduced the complexity of the operator to a finite sum of one-parameter singular integrals. Appealing to Y. Hu and Y. Pan's result [27], since $P(x,y)$ does not contain any linear terms in y, it follows that

$$\int_{\mathbb{R}}\left|\int e^{iP(c,y-t)}\frac{\psi\left(2^{-q}(y-t)\right)}{y-t}b(s,t)\,dt\right| dx \lesssim \int \tilde{U}\,|b(s,t)|\,dt,$$

uniformly in c. Notice that the case $j = j_0$ is treated similarly since we obtain a range $q \sim \log\tilde{c}$. If $i \neq i_0$ and $j \neq j_0$ then not that we may bound $\left\|\sum_{(p,q)\in A_{i,j}^{(1)}} S_{p,q}b\right\|_1$ by

$$M\int\int_{\mathbb{R}^2}\left|\int\int e^{iP(x-s,y-t)}\frac{\psi\left(2^{-f(q)}(x-s)\right)}{x-s}\frac{\psi\left(2^{-q}(y-t)\right)}{y-t}b(s,t)\right| dxdy,$$

which behaves like a one-parameter singular integral in $\mathbb{R}^2$. A little care is needed to adapt Y. Hu and Y. Pan's proof, but it is not hard to see that

$$\int\int_{\mathbb{R}^2}\left|\int\int e^{iP(x-s,y-t)}\frac{\psi\left(2^{-f(q)}(x-s)\right)}{x-s}\frac{\psi\left(2^{-q}(y-t)\right)}{y-t}b(s,t)\right| dxdy \leq C,$$

uniformly in the coefficients of P for any polynomial P.

Remark 5.1. Notice that the region where a monomial dominates the rest tightly is the only case in which we have recurred to the one-parameter theory of Y. Hu and Y. Pan in $\mathbb{R}^2$. In recent progress together with O. Bakas and J. Wright we have managed to avoid the consideration of this region thus allowing P to have linear terms in either variable, say st^2or s^2t.

5.5.2 Region where a monomial largely dominates the rest

We now need to bound

$$\int_{\mathbb{R}^2}\left|\sum_{(p,q)\in A_{i,j}^{(2)}} S_\lambda^{p,q} b\left(x,y\right)\right| dxdy \tag{5.5.2}$$

where $A_{i,j}^{(2)}$ corresponds to the region where the monomial $x^i y^j$ largely dominates the rest, that is

$$A_{i,j}^{(2)} := \left\{ (p,q)\in A_{i,j}, \left|c_{i,j}\right|\left(\left|I\right|2^p\right)^i\left(\left|J\right|2^q\right)^j \geq M \sup_{(m,n)\in\Delta\setminus\{(i,j)\}} \left|c_{m,n}\right|\left(\left|I\right|2^p\right)^m\left(\left|J\right|2^q\right)^n \right\}. \tag{5.5.3}$$

We simplify the notation we rename $A_{i,j}^{(2)} = A$.

An analogous procedure to that detailed in (5.4.3) shows that

$$\int_{\mathbb{R}^2}\left|\sum_{(p,q)\in A} S_\lambda^{p,q} b\left(x,y\right)\right| dxdy \lesssim I_1,$$

where

$$I_1 = \int_{\mathbb{R}^2}\left|\sum_{(p,q)\in A}\int_{\mathbb{R}^2}\left\{e^{i\lambda\tilde{P}(x-s,y-t)} - e^{i\lambda\tilde{P}(x-s,y)}\right\}\frac{\psi\left(2^{-p}\left(x-s\right)\right)}{\left(x-s\right)}\frac{\psi\left(2^{-q}\left(y-t\right)\right)}{\left(y-t\right)} b\left(s,t\right) dsdt\right| dxdy.$$

We thus need to bound the integral $I_1 \leq \sum_{(p,q)\in A} I_{p,q}$, where $I_{p,q}$ corresponds to the integral

$$\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2}\left\{e^{i\lambda\tilde{P}(x-s,y-t)} - e^{i\lambda\tilde{P}(x-s,y)}\right\}\frac{\psi\left(2^{-p}\left(x-s\right)\right)}{x-s}\frac{\psi\left(2^{-q}\left(y-t\right)\right)}{y-t} b\left(s,t\right) dsdt\right| dxdy.$$

Again we will provide several bounds for $I_{p,q}$ and then interpolate between them. The first is analogous to (5.4.6) and is based on an application of the mean value theorem in the second variable of the polynomial P. Note that

$$\left|e^{i\lambda\tilde{P}(x-s,y-t)} - e^{i\lambda\tilde{P}(x-s,y)}\right| \lesssim \lambda\frac{1}{|c|}\left|\tilde{P}\left(x-s,c\right)\right|\left|t\right|,$$

with $|c| \sim |y-t|$, and thus

$$I_{p,q} \lesssim \lambda \sum_{(m,n)\in\Delta} \left|c_{m,n}\right|\left|I\right|^m\left|J\right|^n 2^{pm+q(n-1)}.$$

Further, since $(p,q)\in A_{i,j}$ the monomial (i,j) dominates over the rest (5.5.3),

$$I_{p,q} \lesssim \lambda\left|c_{i,j}\right|\left|I\right|^i\left|J\right|^j 2^{pi+q(j-1)}. \tag{5.5.4}$$

For simplicity of the notation we shall use $\tilde{\lambda} = \lambda\left|c_{i,j}\right|\left|I\right|^i\left|J\right|^j$.

The second estimate will be, as expected, a consequence of the splitting of $I_{p,q}$ into

$$\begin{aligned} I_{p,q}^{(1)} + I_{p,q}^{(2)} = &\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2} e^{i\lambda\tilde{P}(x-s,y-t)}\frac{\psi\left(2^{-p}\left(x-s\right)\right)}{\left(x-s\right)}\frac{\psi\left(2^{-q}\left(y-t\right)\right)}{\left(y-t\right)} b\left(s,t\right) dsdt\right| dxdy \\ &+ \int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2} e^{i\lambda\tilde{P}(x-s,y)}\frac{\psi\left(2^{-p}\left(x-s\right)\right)}{\left(x-s\right)}\frac{\psi\left(2^{-q}\left(y-t\right)\right)}{\left(y-t\right)} b\left(s,t\right) dsdt\right| dxdy. \end{aligned}$$

Again we focus on the study of $I^{(1)}_{p,q}$ and the estimates for $I^{(2)}_{p,q}$ will follow similarly. We define the operator $T^P_{\lambda,p,q}$ as in (5.4.7) for a general polynomial

$$T^P_{\lambda,p,q} g(x,y) = \int_{\mathbb{R}^2} \varphi(s,t)\, e^{i\lambda \tilde{P}(x-s,y-t)} \frac{\psi\left(2^{-p}(x-s)\right)}{(x-s)} \frac{\psi\left(2^{-q}(y-t)\right)}{(y-t)} g(s,t)\, dsdt, \quad (5.5.5)$$

where $\varphi \in C^\infty_c\left(\mathbb{R}^2\right)$ with $\varphi \equiv 1$ on $\left[-\frac{1}{2},\frac{1}{2}\right]^2$ and $\varphi \equiv 0$ outside $[-1,1]^2$. The boundedness properties of $I^{(1)}_{p,q}$ then follow from those of $\left\|T^P_{\lambda,p,q}\right\|_2$, since

$$\sum_{(p,q)\in A} I^{(1)}_{p,q} \le \sum_{(p,q)\in A} 2^{\frac{p+q}{2}} \left\|T^P_{\lambda,p,q} b\right\|_{L^2}.$$

The construction of the region A now plays a crucial role in the proof of the following Lemma.

Lemma 5.3. *Let $T^P_{\lambda,p,q}$ be the operator defined in* (5.5.5) *with* $(p,q)\in A$, *then $T^P_{\lambda,p,q}$ is a bounded operator from $L^2\left(\mathbb{R}^2\right)$ to $L^2\left(\mathbb{R}^2\right)$ and satisfies the following bounds:*

$$\left\|T^P_{\lambda.p,q} b\right\|_{L^2} \lesssim \begin{cases} \frac{1}{2^{\frac{q}{2}}}\left(\frac{1}{\tilde{\lambda}2^{pi+qj}}\right)^{\delta} \|b\|_{L^2}, & p\le 0,\\ \frac{1}{2^{\frac{p+q}{2}}}\left(\frac{1}{\tilde{\lambda}2^{p(i-1)+qj}}\right)^{\delta} \|b\|_{L^2}, & p>0, \end{cases} \quad (5.5.6)$$

for some $0<\delta<1$.

Then, following the definition of $I^{(k)}_{p,q}$, we have

$$I_{p,q} \le \begin{cases} \left(\frac{1}{\tilde{\lambda}2^{pi+qj}}\right)^{\delta}, & p\le 0,\\ \left(\frac{1}{\tilde{\lambda}2^{p(i-1)+qj}}\right)^{\delta}, & p>0. \end{cases} \quad (5.5.7)$$

This Lemma asserts that the operator $T^P_{\lambda,p,q}$ behaves in A as $T^{i,j}_{\lambda,p,q}$in $\tilde{U}$ and will be the carefully proved in the next section. This finishes the proof of Theorem 2.4 as we may combine (5.5.4) and (5.5.6) exactly as in subsections 5.4.1 and 5.4.2 replacing $\tilde{U}$ by A.

5.5.3 $T^*_{\lambda,p,q}T_{\lambda,p,q}$ estimates revisited

In the last section we will be devoted to prove the previous Lemma 5.3. For that purpose we need to produce estimates for the operator $T^P_{\lambda,p,q}$ in the region $(p,q)\in A$. We proceed as in the single monomial case 5.4.3 via estimates of $\left(T^P_{\lambda,p,q}\right)^* T^P_{\lambda,p,q}$. The kernel L of this new operator is

$$L(s,t,u,v) = \varphi(s,t)\,\varphi(u,v)\,\frac{1}{2^{p+q}}$$
$$\int_{(x,y)\in B} e^{i\lambda\left\{\tilde{P}_{p,q}\left(x-2^{-p}s,y-2^{-q}t\right)-\tilde{P}_{p,q}\left(x-2^{-p}u,y-2^{-q}v\right)\right\}} \tilde{G}(x,y,s,t,u,v)\,dxdy,$$

where

$$\tilde{P}_{p,q}(x,y) = \sum_{(i,j)\in\Delta} c_{i,j} 2^{pi+qj} |I|^i |J|^j x^i y^j,$$

$$\tilde{G}(x,y,s,t,u,v) = \frac{\psi(x-2^{-p}s)\,\psi(x-2^{-p}u)\,\psi(y-2^{-q}t)\,\psi(y-2^{-q}v)}{(x-2^{-p}s)(x-2^{-p}u)(y-2^{-q}t)(y-2^{-q}v)},$$

and $B = \left\{(x,y)\in\mathbb{R}^2, |x-2^{-p}s| \sim |x-2^{-p}u| \sim |y-2^{-p}t| \sim |y-2^{-p}v| \sim 1\right\}$.

Therefore as in (5.4.15) the following estimates are enough to show the L^2 boundedness of $T^P_{\lambda,p,q}$(5.5.6):

$$\sup_{u,v}\int |L(s,t,u,v)|\,dsdt \lesssim \begin{cases} \frac{1}{2^q}\left(\frac{1}{\bar{\lambda}2^{pi+qj}}\right)^\delta, & p\le 0,\\ \frac{1}{2^q}\left(\frac{1}{\bar{\lambda}2^{p(i-1)+qj}}\right)^\delta, & p>0,\end{cases} \tag{5.5.8}$$

and

$$\sup_{s,t}\int |L(s,t,u,v)|\,dudv \lesssim \begin{cases} \frac{1}{2^q}\left(\frac{1}{\bar{\lambda}2^{pi+qj}}\right)^\delta, & p\le 0,\\ \frac{1}{2^q}\left(\frac{1}{\bar{\lambda}2^{p(i-1)+qj}}\right)^\delta, & p>0.\end{cases}$$

We denote by ϕ the oscillatory phase

$$\phi(x,y) = \lambda\left\{\tilde{P}_{p,q}\left(x-2^{-p}s, y-2^{-q}t\right) - \tilde{P}\left(x-2^{-p}u, y-2^{-q}v\right)\right\},$$

and look for upper bounds of derivatives of the oscillatory phase in order to apply the n-dimensional *van der Corput lemma*, Proposition A.5. In fact we are interested in the $i+j-1$ derivative

$$\frac{\partial^{i+j-1}\phi}{\partial x^{i-1}\partial y^j}(x,y) = \sum_{(m,n)\in\Delta}\frac{\partial^{i+j-1}\phi_{m,n}}{\partial x^{i-1}\partial y^j}(x,y)$$
$$\sim \sum_{\substack{m\ge i-1\\ n\ge j}} C_{m,n}\left\{\left(x-2^{-p}s\right)^{m-i+1}\left(y-2^{-q}t\right)^{n-j} - \left(x-2^{-p}u\right)^{m-i+1}\left(y-2^{-q}v\right)^{n-j}\right\},$$

where

$$C_{m,n} = \lambda c_{m,n}\,|I|^m\,|J|^n\,2^{pm+qn}.$$

For all $m\ge i-1, n\ge j$ we have that $\frac{\partial^{i+j-1}\phi_{m,n}}{\partial x^{i-1}\partial y^j}(x,y)$ is equal to

$$C_{m,n}\left\{\left(x-2^{-p}s\right)^{m-i+1}\left(y-2^{-q}t\right)^{n-j} - \left(x-2^{-p}u\right)^{m-i+1}\left(y-2^{-q}v\right)^{n-j}\right\}$$

Then, we rewrite the quantity above as

$$C_{m,n}\left\{\left(x-2^{-p}s\right)^{m-i+1} - \left(x-2^{-p}u\right)^{m-i+1}\right\}\left(y-2^{-q}t\right)^{n-j}$$
$$+ C_{m,n}\left(x-2^{-p}u\right)^{m-i+1}\left\{\left(y-2^{-q}t\right)^{n-j} - \left(y-2^{-q}v\right)^{n-j}\right\}.$$

That is we have

$$\frac{\partial^{i+j-1}\phi_{m,n}}{\partial x^{i-1}\partial y^j}(x,y) = \sum_{\substack{m\ge i\\ n\ge j}} A_{m,n} + \sum_{\substack{m\ge i-1\\ n\ge j+1}} B_{m,n},$$

and using the mean value theorem together with the fact that $(x, y) \in B$,

$$A_{m,n} = C_{m,n} 2^{-p} (u - s) \alpha_n \beta_m$$

and

$$B_{m,n} = C_{m,n} 2^{-q} (v - t) \gamma_m \delta_n,$$

where $|\alpha_n| \sim |\gamma_m| \sim |\beta_m| \sim |\delta_n| \sim 1$ for all (m, n). Therefore

$$|A_{m,n}| \sim |C_{m,n}| \, 2^{-p} \, |u - s|$$

and

$$|B_{m,n}| \sim |C_{m,n}| \, 2^{-q} \, |v - t| \, .$$

In order to give an upper bound of $\left|\sum \frac{\partial^{i+j-1}\phi_{m,n}}{\partial x^{i-1}\partial y^j}\right|$ we will make use of the comparison principle given by (5.5.3):

$$|C_{i,j}| \geq 2M \sup_{(i,j)\in\Delta\setminus\{(m,n)\}} |C_{m,n}|, \tag{5.5.9}$$

where M is as big as we want, as long as it only depends on $|\Delta|$. Using the reverse triangle inequality, we have

$$\left|\sum \frac{\partial^{i+j-1}\phi_{m,n}}{\partial x^{i-1}\partial y^j}(x, y)\right| \gtrsim \left|\left|\sum_{\substack{m\geq i\\ n\geq j}} A_{m,n} C_{i,j} 2^{-p} (u - s)\right| - \left|\sum_{\substack{m\geq i-1\\ n\geq j+1}} (n - j) \, C_{m,n} 2^{-q} (v - t)\right|\right|.$$

First note that given (5.5.9) it follows that

$$\left|\sum_{\substack{m\geq i\\ n\geq j}} C_{m,n} 2^{-p} (u - s) \, \alpha_n \, (m - i + 1) \, \beta_m\right| \sim \frac{1}{2} |C_{i,j}| \, 2^{-p} \, |u - s| \, . \tag{5.5.10}$$

Bare in mind that our objective is to show that the estimates (5.5.8) hold. Let us take care only of the case $p \leq 0$ as the other one will be similarly obtained. Since $p \leq 0$ note that $L = 0$ unless $|u - s| < 2^p$. Then

$$\sup_{u,v} \int |L(s, t, u, v)| \, dsdt = \sup_{u,v} \sum_{l=1}^{3} \int_{\substack{(s,t)\in B_l\\ |u-s|<2^p}} |L(s, t, u, v)| \, dsdt,$$

where the regions B_l are defined as follows:

$$\begin{aligned}
B_1 &= \left\{(s, t) \in [-1, 1]^2, |C_{i,j}| \, 2^{-p} \, |u - s| > 2 \left|\sum B_{m,n}\right|\right\}, \\
B_2 &= \left\{(s, t) \in [-1, 1]^2, \left|\sum B_{m,n}\right| > 2 \, |C_{i,j}| \, 2^{-p} \, |u - s|\right\}, \\
B_3 &= \left\{(s, t) \in [-1, 1]^2, |C_{i,j}| \, 2^{-p} \, |u - s| \sim \left|\sum B_{m,n}\right|\right\}.
\end{aligned}$$

Then, using (5.5.10) and the n-dimensional *van der Corput Lemma*, we obtain

$$\begin{aligned}
\sup_{u,v}\int_{B_1}|L(s,t,u,v)|\,dsdt &\lesssim \sup_{u,v}\frac{1}{2^{p+q}}\left(\frac{1}{2^p|C_{i,j}|}\right)^{\frac{1}{i+j-1}}\int_{|u-s|<2^p}|u-s|^{-\frac{1}{i+j-1}}\,ds\\
&\leq \frac{2^{p\left(1-\frac{1}{i+j-1}\right)}}{2^{p+q}}\left(\frac{1}{2^p|C_{i,j}|}\right)^{\frac{1}{i+j-1}}\\
&\leq \frac{1}{2^q}\left(\frac{1}{\tilde{\lambda}2^{pi+qj}}\right)^{\frac{1}{i+j-1}}. \qquad (5.5.11)
\end{aligned}$$

On the other hand, note that in B_2,

$$\begin{aligned}
\left|\sum\frac{\partial^{i+j-1}\phi_{m,n}}{\partial x^{i-1}\partial y^j}\right| &\gtrsim \left|\sum_{n\geq j}B_{m,n}\right|\\
&\gtrsim |C_{i,j}|\,2^{-p}\,|u-s|,
\end{aligned}$$

and thus

$$\begin{aligned}
\sup_{u,v}\int_{B_2}|L(s,t,u,v)|\,dsdt &\lesssim \sup_{u,v}\frac{1}{2^{p+q}}\left(\frac{1}{2^{-p}|C_{i,j}|}\right)^{\frac{1}{i+j-1}}\int_{|u-s|<2^p}|u-s|^{-\frac{1}{i+j-1}}\,ds\\
&\lesssim \frac{1}{2^q}\left(\frac{1}{\tilde{\lambda}2^{pi+qj}}\right)^{\frac{1}{i+j-1}}. \qquad (5.5.12)
\end{aligned}$$

Finally, if $(s,t)\in B_3$ we will make use of one more derivative in x and one less in y since the polynomial P does not contain terms in either variable. That is we compute

$$\frac{\partial^{i+j-1}\phi_{m,n}}{\partial x^i\partial y^{j-1}}(x,y) \sim \sum_{\substack{m\geq i+1\\ n\geq j-1}}\tilde{A_{m,n}}+\sum_{\substack{m\geq i\\ n\geq j}}\tilde{B_{m,n}}$$

where

$$\left|\tilde{A_{m,n}}\right|\sim|A_{m,n}| \quad\text{and}\quad \left|\tilde{B_{m,n}}\right|\sim|B_{m,n}|.$$

Let us highlight that a new term $\tilde{B_{i,j}}$ appears in the sum and that it largely dominates the rest. This follows once more from the comparison principle (5.5.9) and the restriction $(x,y)\in B_3$, as

$$\left|\tilde{B_{i,j}}\right|\sim|C_{i,j}|\,2^{-q}\,|v-t|\geq 2M\sup_{(m,n)\in\Delta\setminus\{(i,j)\}}|B_{m,n}|.$$

Therefore

$$\left|\tilde{B_{i,j}}\right|\sim\left|\sum_{\substack{m\geq i-1\\ n\geq j+1}}B_{m,n}\right|\sim|C_{i,j}|\,2^{-p}\,|u-s|$$

and

$$\left|\tilde{B_{i,j}}\right|\sim\left|\sum_{\substack{m\geq i\\ n\geq j}}\tilde{B_{m,n}}\right|.$$

Hence, we have the estimate

$$\left|\frac{\partial^{i+j-1}\phi_{m,n}}{\partial x^i \partial y^{j-1}}(x,y)^{i+j-1}\right| \geq \frac{1}{2}\left|C_{i,j}\right| 2^{-p}\left|u-s\right|$$

and we proceed as usual:

$$\begin{aligned} \sup_{u,v}\int_{B_2}\left|L\left(s,t,u,v\right)\right| dsdt &\lesssim \sup_{u,v}\frac{1}{2^{p+q}}\left(\frac{1}{2^{-p}\left|C_{i,j}\right|}\right)^{\frac{1}{i+j}}\int_{|u-s|<2^p}\left|u-s\right|^{-\frac{1}{i+j}} ds \\ &\lesssim \frac{2^{p\left(1-\frac{1}{i+j}\right)}}{2^{p+q}}\left(\frac{1}{2^{-p}\left|C_{i,j}\right|}\right)^{\frac{1}{i+j}} \\ &\lesssim \frac{1}{2^q}\left(\frac{1}{\tilde{\lambda}2^{pi+qj}}\right)^{\frac{1}{i+j-1}}. \end{aligned} \tag{5.5.13}$$

Finally, combining the estimates (5.5.11), (5.5.12) and (5.5.13) we have that

$$\sup_{u,v}\int\left|L\left(s,t,u,v\right)\right| dsdt \lesssim \frac{1}{2^q}\left(\frac{1}{\tilde{\lambda}2^{pi+qj}}\right)^{\delta},$$

for some $\delta > 0$. Similarly we get

$$\sup_{s,t}\int\left|L\left(s,t,u,v\right)\right| dudv \lesssim \frac{1}{2^q}\left(\frac{1}{\tilde{\lambda}2^{pi+qj}}\right)^{\delta},$$

obtaining the needed estimates to apply *Schur's lemma.* An analogous procedure can be carried out on the case $p > 0$, noting that now $L = 0$ unless $|u-s| < 1$.

Finally note that throughout the proof of Theorem 2.4 the only bound that depends in the coefficients of the polynomial corresponds to the L^2 bound (5.3.1). Therefore, by imposing the more restrictive condition that P is in $\mathcal{P}_{even}$, this bound will also be uniform in the coefficients of the polynomial and we obtain Corollary 2.3

Appendix A

Oscillatory integrals of the first kind

First of all let us mention that the contents of this section are heavily based on Chapter VIII of [40]. We will not present any proof of the results mentioned, but we will rather concentrate on their applications to some examples.

Consider the oscillatory integral of one variable

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\psi(x)\,dx,$$

where ϕ is a real-valued smooth function (the *phase*) and ψ is a complex-valued smooth function with compact support in (a, b).

Heuristically, if the phase ϕ oscillates, that is $\phi'(x) \neq 0$ for all $x \in [a, b]$, the integral will profit from a lot of cancellation. Furthermore, this cancellation will be increased as $\lambda \to \infty$ forcing $|I(\lambda)|$ to rapidly decrease. This is phenomena is called the *localization* principle:

Proposition A.1. *Let ϕ and ψ be smooth functions so that ψ has compact support in (a, b), and $\phi'(x) \neq 0$ for all $x \in [a, b]$. Then*

$$I(\lambda) = O\left(\lambda^{-N}\right) \text{ as } \lambda \to \infty,$$

for all $N \geq 0$.

The second principle underlying the asymptotic behaviour of oscillatory integrals is known as *scaling.* It quantifies the decay of $|I(\lambda)|$ in therms of estimates on the size of the derivatives of ϕ. The following estimate goes back to van der Corput and is generally referred to as *van der Corput estimates:*

Proposition A.2. *Suppose ϕ is real-valued and smooth in (a, b), and that $\left|\phi^{(k)}(x)\right| \geq 1$ for all $x \in (a, b)$. Then*

$$\left|\int_a^b e^{i\lambda\phi(x)}dx\right| \leq c_k\lambda^{-\frac{1}{k}}$$

holds when :

1. $k \geq 2$, *or*

2. $k = 1$ *and* $\phi'(x)$ *is monotonic.*

The bound c_k *is independent of* ϕ *and* λ.

The localization principle asserts that the behaviour of $I(\lambda)$ is determined by the *critical* points $x_0 \in [a, b]$ where the phase behaves stationary, that is $\phi'(x_0) = 0$. The third principle goes further on this direction and asserts that if there is only one critical point in the support of ψ then the asymptotic expansion of $I(\lambda)$ depends on the smallest $k \geq 2$ such that $\phi^{(k)}(x_0) \neq 0$. This phenomena is known as the *stationary phase method:*

Proposition A.3. *Suppose* $k \geq 2$, *and*

$$\phi^{(j)}(x_0) = 0,$$

for all $j < k$, *while* $\phi^{(k)}(x_0) \neq 0$. *If* ψ *is supported in a sufficiently small neighbourhood of* x_0, *then*

$$I(\lambda) \sim \lambda^{-\frac{1}{k}} \sum_{j=0}^{\infty} a_j \lambda^{-\frac{j}{k}},$$

in the sense that, for all nonnegative integers N *and* r,

$$\left(\frac{d}{d\lambda}\right)^r \left[I(\lambda) - \lambda^{-\frac{1}{k}} \sum_{j=0}^{N} a_j \lambda^{-\frac{j}{k}}\right] = O\left(\lambda^{-r-(N+1)/k}\right) \text{ as } \lambda \to \infty.$$

In higher dimensions only some of these principles have analogues. Consider the following oscillatory integral

$$I(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x)} \psi(x)\, dx,$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ is a smooth real valued function and $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is smooth with compact support. We say that the phase ϕ has a critical point x_0 if

$$\nabla\phi(x_0) = \left(\frac{\partial\phi}{\partial x_1}, \ldots, \frac{\partial\phi}{\partial x_n}\right)\Big|_{x=x_0} = 0.$$

The localization principle holds the same way as in the one dimensional case.

Proposition A.4. *Suppose* ψ *is smooth, has compact support and that* ϕ *is a smooth real-valued function that has no critical points in the support of* ψ. *Then*

$$I(\lambda) = O\left(\lambda^{-N}\right)$$

as $\lambda \to \infty$, *for every* $N \geq 0$.

However, the scaling principle only admits weak analogues.

Proposition A.5. *Suppose ψ is smooth and is supported in the unit ball; also let ϕ be a real valued function so that, for some multi-index α with $|\alpha| > 0$, we have*

$$|\partial_x^\alpha \phi| \geq 1$$

throughout the support of ψ. Then

$$|I(\lambda)| \leq c_k(\phi)\, \lambda^{-\frac{1}{k}} \left(\|\psi\|_\infty + \|\nabla\psi\|_{L^1} \right),$$

where $k = |\alpha|$; the constant $c_k(\phi)$ is independent of λ and ψ, and remains bounded as long as the C^{k+1} norm of ϕ remains bounded.

Let us highlights that this expression admits and extension in the case where ϕ is a polynomial and the support of ψ is included in $[0,1]^n$. In this special case the constant c_k depends only on the degree of the polynomial, but not in its coefficients.

The stationary phase method also admits an extension to the higher dimensional analogues. Suppose that ϕ has a critical point at x_0 and that the symmetric $n \times n$ matrix

$$\left[\frac{\partial^2 \phi}{\partial x_i \partial x_j} \right] (x_0)$$

is invertible, then the critical point is said to be *nondegenerate.*

Proposition A.6. *Suppose $\phi(x_0) = 0$, and ϕ has a nondegenerate critical point at x_0. If ψ is supported in a sufficiently small neighbourhood of x_0, then*

$$\int_{\mathbb{R}^n} e^{i\lambda\phi(x)} \psi(x)\, dx \sim \lambda^{-\frac{n}{2}} \sum_{j=0}^{\infty} a_j \lambda^{-j}, \text{ as } \lambda \to \infty,$$

with the asymptotics in the same sense as in Proposition A.3.

In the next sections we will apply the ideas just presented to oscillatory integrals that appear throughout this thesis.

A.1 Bessel functions

Bessel functions $J_n(x)$ arise naturally as a family of solutions of Bessel's equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \left(x^2 - n^2\right) y = 0.$$

For integers they are defined by means of the Laurent's expansion of

$$e^{\frac{1}{2}x\left(z - \frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} z^n J_n(x),$$

for $z \in \mathbb{C}\backslash\{0\}$ and $x \geq 0$, and may be expressed by means of its Poisson representation formula

$$J_n(x) = \frac{\left(\frac{x}{2}\right)^n}{\Gamma\left(n + \frac{1}{2}\right)\sqrt{\pi}} \int_{-1}^{1} e^{ixt} \left(1 - t^2\right)^{n - \frac{1}{2}} dt.$$

In fact the the definition of Bessel functions may be extended by means of the Poisson representation formula to all real orders $\nu > -\frac{1}{2}$.

From this expression it is easy to check that

$$J_\nu(x) = \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(n+1)}, \text{ as } x \to 0.$$

To obtain an asymptotic of $J_\nu(x)$ as $x \to \infty$ is a more complicated affair and there exist several procedures. One of them relies on Cauchy's Theorem and the identity for complex order ν with $\Re\nu > -1/2$,

$$\int_{-1}^{1} e^{ixt}\left(1-t^2\right)^{\nu-\frac{1}{2}} dt = \\ ie^{-ix}\left\{\int_0^\infty e^{-xt}\left(t^2+2it\right)^{\nu-\frac{1}{2}} dt - \int_0^\infty e^{-xt}\left(t^2-2it\right)^{\nu-\frac{1}{2}} dt\right\},$$

which combined with the Poisson representation formula yields

$$|J_\nu(x)| \leq c(\nu)\, x^{-\frac{1}{2}}.$$

It is however a more delicate matter to understand the behaviour of $J_\nu(x)$ when $x \sim \nu$. This study was carried by J. A. Barceló and A. Córdoba [2] who produced the following estimates:

Lemma A.1. *The following estimates hold for $\nu \geq 1$.*

1. $|J_\nu(r)| \leq \frac{1}{r^{1/2}}$ *and* $|J'_\nu(r)| \leq \frac{1}{r^{1/2}}$ *when* $r \geq 2\nu$.
2. $|J_\nu(r)| \leq \frac{1}{\nu}$, *and* $|J'_\nu(r)| \leq \frac{1}{(1+\nu)^2}$ *when* $r \leq \frac{1}{2}\nu$.
3. $\left|J_\nu\left(\nu+\rho\nu^{1/3}\right)\right| \leq \frac{1}{\rho^{1/4}\nu^{1/3}}$, *and* $\left|J'_\nu\left(\nu+\rho\nu^{1/3}\right)\right| \leq \frac{1}{\rho^{1/4}\nu^{2/3}}$, *when* $0 \leq \rho \leq \nu^{\frac{2}{3}}$.
4. $\left|J_\nu\left(\nu-\rho\nu^{1/3}\right)\right| \leq \frac{1}{\rho\nu^{1/3}}$, *and* $\left|J'_\nu\left(\nu-\rho\nu^{1/3}\right)\right| \leq \frac{1}{\rho^2\nu^{2/3}}$, *when* $1 \leq \rho \leq \frac{1}{2}\nu^{\frac{2}{3}}$.
5. $|J_\nu(r)| \leq r^\nu$, *as* $r \to 0$.

Proof. These estimates are based on van der Corput estimates and the stationary phase method. We use a different expression of Bessel functions,

$$J_\nu(x) = \frac{1}{2\pi}\Re\int_{-\pi}^{\pi} e^{i(\nu\theta - x\sin\theta)}d\theta - \frac{\sin(\pi\nu)}{\pi}\int_0^\infty e^{-x\sinh t-\nu t}dt,$$

for all real $\nu > -\frac{1}{2}$ and $x \geq 0$. This way we can see Bessel functions as the sum of a principal oscillatory term and an error. Indeed, notice that for any $x > 0$, trivially

$$\left|\frac{\sin(\pi\nu)}{\pi}\int_0^\infty e^{-x\sinh t-\nu t}dt\right| \lesssim \frac{1}{\nu+x}.$$

Therefore we may concentrate our efforts in the first term of $J_\nu(x)$

$$I_\nu(x) = \int_{-\pi}^{\pi} e^{i(\nu\theta - x\sin\theta)}d\theta$$

which corresponds to a singular integral.

Remark that we have already hinted the proofs of several of the desired estimates and thus we will only show 3) as 4) follows similarly. We thus focus in the region where $\frac{1}{2}\nu \leq x \leq 2\nu$. Note that $I_\nu(x)$ only has one nondegenerate critical point $\theta_0 = \cos^{-1}(\nu/x)$, and that

$$\frac{d^2}{d\theta^2}(n\theta - x\sin\theta)\Big|_{\theta_0} = x\sqrt{1-\left(\frac{\nu}{x}\right)^2}.$$

Further, note that we want to obtain an estimate when $x \sim \nu + \rho\nu^1$ and thus $\nu/x \sim 1 - \nu^{-\frac{2}{3}}$. We may thus obtain an upper bound for the second derivative of the phase

$$\frac{d^2}{d\theta^2}(n\theta - x\sin\theta)\Big|_{\theta_0} \sim \rho^{\frac{1}{2}}\nu^{\frac{2}{3}}.$$

Finally applying van der Corput's estimate, we obtain

$$|J_\nu(x)| \lesssim \frac{1}{\rho^{\frac{1}{4}}\nu^{\frac{1}{3}}},$$

if $x \sim \nu + \rho\nu^{\frac{1}{3}}$ and $1 < \rho < \nu^{\frac{2}{3}}$.

In order to produce the estimates for $J'_\nu(x)$ we proceed analogously, but notice that an integration by parts yields that $J'_\nu(x)$ is the real part of

$$\frac{1}{2\pi i}\int_{-\pi}^{\pi}\sin\theta e^{i(\nu\theta - x\sin\theta)}d\theta - \frac{\sin(\pi\nu)}{\pi}\int_0^\infty e^{-x\sinh t - \nu t}dt.$$

The extra oscillatory term $\sin\theta$ is the cause that we are able to obtain some extra decay in ν on all estimates. □

A.2 Generalized Hilbert Transform kernels

Another application of the oscillatory integral principles is found in the study of the one dimensional Hilbert transforms with an oscillatory phase. As stated in the introduction, it is of great interest to obtain a uniform bound of the oscillatory integral

$$\left|\int \frac{e^{iP(s)}}{s}ds\right|,$$

for any polynomial P. We follow the slightly simplified approach presented by S. Wainger in [49].

Proof. We want to show that

$$A = \left|\int_{\epsilon<|s|<R}\frac{e^{iP(s)}}{s}ds\right| < c$$

uniformly in ϵ, R and in the coefficients of $\tilde{P}$. Notice that we may assume without loss of generality that $P(s) = a_1 s + a_2 s^2 + \ldots \pm s^n$.

We first decompose A into two regions

$$A = \int_{\epsilon<|s|<1} + \int_{1<|s|<R}$$

and recall that *van der Corput's Lemma* (see Proposition A.2) yields the following bound

$$\int_1^t e^{iP(s)}ds < c(n), \tag{A.2.1}$$

since the highest degree coefficient of P is ± 1. Also, let

$$F(t) := \int_1^t e^{iP(s)}ds,$$

so that $F'(t) = e^{iP(t)}$. Then rewrite

$$\begin{aligned} \left| \int_{1<|s|<R} \frac{e^{iP(s)}}{s} ds \right| &= \left| \int_{1<|s|<R} \frac{F'(s)}{s} ds \right| \\ &\leq \left| \left[\frac{F(s)}{s} \right]_1^R \right| + \left| \int_{1<|s|<R} \frac{F(s)}{s^2} ds \right| \end{aligned}$$

and use (A.2.1). Hence

$$\left| \int_{1<|s|<R} \frac{e^{iP(s)}}{s} ds \right| \leq c(n) \left(1 + \int_{1<|s|<R} \frac{1}{s^2} ds \right)$$

In order to control the integral over the region near the origin we proceed by induction on the degree n of the polynomial. Let $Q(s) = a_1 s + a_2 s^2 + \ldots + a_{n-1} s^{n-1}$ and assume that

$$\left| \int_{\epsilon<|s|<1} \frac{e^{iQ(s)}}{s} ds \right| < c(n),$$

uniformly in ϵ, R and in the coefficients of Q. Then

$$\begin{aligned} \left| \int_{\epsilon<|s|<1} \frac{e^{iP(s)}}{s} ds \right| &\leq \left| \int_{\epsilon<|s|<1} \frac{e^{iQ(s)}}{s} ds \right| + \left| \int_{\epsilon<|s|<1} e^{iQ(s)} \left(e^{it^n} - 1 \right) \frac{ds}{s} \right| \\ &\leq \int_{\epsilon<|s|<1} \left| e^{it^n} - 1 \right| \frac{ds}{|s|}. \end{aligned}$$

It is a simple matter to check that the *Mean Value Theorem* yields $\left| e^{it^n} - 1 \right| \leq n |t|^{2n-1}$, and thus

$$\begin{aligned} \int_{\epsilon<|s|<1} \left| e^{it^n} - 1 \right| \frac{ds}{|s|} &\leq n \int_{\epsilon<|s|<1} |t|^{2n-2} ds \\ &\leq 1, \end{aligned}$$

since $n \geq 1$. We have thus shown that

$$\left| \int_{\epsilon<|s|<1} \frac{e^{iP(s)}}{s} ds \right| \leq 1 + \left| \int_{\epsilon<|s|<1} \frac{e^{iQ(s)}}{s} ds \right|.$$

Iterating this procedure we find that

$$\left| \int_{\epsilon<|s|<1} \frac{e^{iP(s)}}{s} ds \right| \leq n + \left| \int_{\epsilon<|s|<1} \frac{e^{is}}{s} ds \right|.$$

We follow the standard argument that exploits the cancellation $\int_{\epsilon<|s|<1} \frac{1}{s} ds = 0$ to finish to show that

$$\left| \int_{\epsilon<|s|<1} \frac{e^{is}}{s} ds \right| \leq 2$$

and finish the proof. □

Note that this proof cannot be generalized to higher dimensions. This was achieved much later by F. Ricci and E. M. Stein in [36]. In this setting some subtle necessary extra cancellation is needed to bound the integral uniformly in the coefficients. This condition may, in turn, be viewed as geometrical constrains in the polynomial phase. Still today this research field remains active, [6].

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